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Free Many Sorted Universal Algebra

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The terminology and notation used in this paper are introduced in the following papers: [21], [24], [25], [11], [22], [12], [7], [18], [13], [10], [2], [4], [5], [23], [14], [6], [1], [16], [3], [8], [20], [17], [19], [9], and [15].

1. Preliminaries

The following proposition is true

(1) Let I be a set, and let J be a non empty set, and let f be a function from I into J^*, and let X be a many sorted set of J, and let p be an element of J^*, and let x be arbitrary. If $x \in I$ and $p = f(x)$, then $(X^{\#} \cdot f)(x) = \prod(X \cdot p)$.

Let I be a set, let A, B be many sorted sets of I, let C be a many sorted subset of A, and let F be a many sorted function from A into B. The functor $F \restriction C$ yielding a many sorted function from C into B is defined as follows:

(Def.1) For arbitrary i such that $i \in I$ and for every function f from $A(i)$ into $B(i)$ such that $f = F(i)$ holds $(F \restriction C)(i) = f \restriction C(i)$.

Let I be a set, let X be a many sorted set of I, and let i be arbitrary. Let us assume that $i \in I$. The functor $\mathrm{coprod}(i, X)$ yields a set and is defined as follows:

(Def.2) For arbitrary x holds $x \in \mathrm{coprod}(i, X)$ iff there exists arbitrary a such that $a \in X(i)$ and $x = \langle a, i \rangle$.

Let I be a set and let X be a many sorted set of I. Then $\mathrm{disjoint}\, X$ is a many sorted set of I and it can be characterized by the condition:

(Def.3) For arbitrary i such that $i \in I$ holds $(\mathrm{disjoint}\, X)(i) = \mathrm{coprod}(i, X)$.

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We introduce coprod(X) as a synonym of disjoint X.

Let I be a non empty set and let X be a non-empty many sorted set of I. One can verify that coprod(X) is non-empty.

Let I be a non empty set and let X be a non-empty many sorted set of I. One can check that $\bigcup X$ is non empty.

We now state the proposition

(2) Let I be a set, and let X be a many sorted set of I, and let i be arbitrary. If $i \in I$, then $X(i) \neq \emptyset$ iff $(\text{coprod}(X))(i) \neq \emptyset$.

2. Free Many Sorted Universal Algebra - General Notions

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. A subset of U_0 is said to be a generator set of U_0 if:

(Def.4) The sorts of Gen(it) = the sorts of U_0.

Next we state the proposition

(3) Let S be a non void non empty many sorted signature, and let U_0 be a strict non-empty algebra over S, and let A be a subset of U_0. Then A is a generator set of U_0 if and only if $\text{Gen}(A) = U_0$.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. A generator set of U_0 is free if it satisfies the condition (Def.5).

(Def.5) Let U_1 be a non-empty algebra over S and let f be a many sorted function from it into the sorts of U_1. Then there exists a many sorted function h from U_0 into U_1 such that h is a homomorphism of U_0 into U_1 and $h \restriction \text{it} = f$.

Let S be a non void non empty many sorted signature. A non-empty algebra over S is free if:

(Def.6) There exists generator set of it which is free.

The following proposition is true

(4) Let S be a non void non empty many sorted signature and let X be a many sorted set of the carrier of S. Then $\bigcup \text{coprod}(X) \cap [\!: \text{the operation symbols of } S, \{\text{the carrier of } S\} :\!] = \emptyset$.

3. Semidisjoint Many Sorted Signature

Let S be a non void many sorted signature. Note that the operation symbols of S is non empty.

Let S be a non void non empty many sorted signature and let X be a many sorted set of the carrier of S. The functor REL(X) yields a relation between $[\!: \text{the operation symbols of } S, \{\text{the carrier of } S\} :\!] \cup \bigcup \text{coprod}(X)$ and

$([\!:$ the operation symbols of S, $\{$the carrier of $S\}:\!] \cup \bigcup \mathrm{coprod}(X))^*$ and is defined by the condition (Def.9).

(Def.9)[1] Let a be an element of $[\!:$the operation symbols of S, $\{$the carrier of $S\}:\!] \cup \bigcup \mathrm{coprod}(X)$ and let b be an element of $([\!:$the operation symbols of S, $\{$the carrier of $S\}:\!] \cup \bigcup \mathrm{coprod}(X))^*$. Then $\langle a, b\rangle \in \mathrm{REL}(X)$ if and only if the following conditions are satisfied:

(i) $a \in [\!:$the operation symbols of S, $\{$the carrier of $S\}:\!]$, and

(ii) for every operation symbol o of S such that $\langle o$, the carrier of $S\rangle = a$ holds $\mathrm{len}\, b = \mathrm{len}\, \mathrm{Arity}(o)$ and for arbitrary x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [\!:$the operation symbols of S, $\{$the carrier of $S\}:\!]$, then for every operation symbol o_1 of S such that $\langle o_1$, the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 = \mathrm{Arity}(o)(x)$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then $b(x) \in \mathrm{coprod}(\mathrm{Arity}(o)(x), X)$.

In the sequel S will be a non void non empty many sorted signature, X will be a many sorted set of the carrier of S, o will be an operation symbol of S, and b will be an element of $([\!:$the operation symbols of S, $\{$the carrier of $S\}:\!] \cup \bigcup \mathrm{coprod}(X))^*$.

Next we state the proposition

(5) $\langle\langle o$, the carrier of $S\rangle, b\rangle \in \mathrm{REL}(X)$ if and only if the following conditions are satisfied:

(i) $\mathrm{len}\, b = \mathrm{len}\, \mathrm{Arity}(o)$, and

(ii) for arbitrary x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [\!:$the operation symbols of S, $\{$the carrier of $S\}:\!]$, then for every operation symbol o_1 of S such that $\langle o_1$, the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 = \mathrm{Arity}(o)(x)$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then $b(x) \in \mathrm{coprod}(\mathrm{Arity}(o)(x), X)$.

Let S be a non void non empty many sorted signature and let X be a many sorted set of the carrier of S. The functor DTConMSA(X) yielding a strict tree construction structure is defined as follows:

(Def.10) $\mathrm{DTConMSA}(X) = \langle [\!:$the operation symbols of S, $\{$the carrier of $S\}:\!] \cup \bigcup \mathrm{coprod}(X), \mathrm{REL}(X)\rangle$.

Let S be a non void non empty many sorted signature and let X be a many sorted set of the carrier of S. Observe that DTConMSA(X) is non empty.

We now state the proposition

(6) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Then the nonterminals of $\mathrm{DTConMSA}(X) = [\!:$the operation symbols of S, $\{$the carrier of $S\}:\!]$ and the terminals of $\mathrm{DTConMSA}(X) = \bigcup \mathrm{coprod}(X)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Observe that DTConMSA(X) has terminals, nonterminals, and useful nonterminals.

One can prove the following proposition

[1]The definitions (Def.7) and (Def.8) have been removed.

(7) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let t be arbitrary. Then $t \in$ the terminals of DTConMSA(X) if and only if there exists a sort symbol s of S and there exists arbitrary x such that $x \in X(s)$ and $t = \langle x, s\rangle$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, and let o be an operation symbol of S. The functor Sym(o, X) yielding a symbol of DTConMSA(X) is defined by:

(Def.11) Sym$(o, X) = \langle o$, the carrier of $S\rangle$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, and let s be a sort symbol of S. The functor FreeSort(X, s) yielding a non empty subset of TS(DTConMSA(X)) is defined by the condition (Def.12).

(Def.12) FreeSort$(X, s) = \{a : a$ ranges over elements of TS(DTConMSA(X)), $\bigvee_x\ x \in X(s)\ \wedge\ a =$ the root tree of $\langle x, s\rangle\ \vee\ \bigvee_o\ \langle o$, the carrier of $S\rangle = a(\varepsilon)\ \wedge$ the result sort of $o = s\}$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. The functor FreeSorts(X) yielding a non-empty many sorted set of the carrier of S is defined by:

(Def.13) For every sort symbol s of S holds (FreeSorts(X))(s) = FreeSort(X, s).

The following propositions are true:

(8) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let o be an operation symbol of S, and let x be arbitrary. Suppose $x \in$ ((FreeSorts(X))$^{\#}$ $\cdot$ (the arity of S))(o). Then x is a finite sequence of elements of TS(DTConMSA(X)).

(9) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let o be an operation symbol of S, and let p be a finite sequence of elements of TS(DTConMSA(X)). Then $p \in$ ((FreeSorts(X))$^{\#}$ $\cdot$ (the arity of S))(o) if and only if $\operatorname{dom} p = \operatorname{dom} \operatorname{Arity}(o)$ and for every natural number n such that $n \in \operatorname{dom} p$ holds $p(n) \in$ FreeSort$(X, \pi_n \operatorname{Arity}(o))$.

(10) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let o be an operation symbol of S, and let p be a finite sequence of elements of TS(DTConMSA(X)). Then Sym$(o, X) \Rightarrow$ the roots of p if and only if $p \in$ ((FreeSorts(X))$^{\#}$ $\cdot$ (the arity of S))(o).

(11) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let o be an operation symbol of S. Then (FreeSorts(X) $\cdot$ (the result sort of S))(o) $\neq \emptyset$.

(12) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Then $\bigcup \operatorname{rng}$ FreeSorts(X) = TS(DTConMSA(X)).

(13) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let s_1, s_2 be sort symbols of S. If $s_1 \neq s_2$, then $(\mathrm{FreeSorts}(X))(s_1) \cap (\mathrm{FreeSorts}(X))(s_2) = \emptyset$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, and let o be an operation symbol of S. The functor $\mathrm{DenOp}(o, X)$ yielding a function from $((\mathrm{FreeSorts}(X))^{\#} \cdot (\text{the arity of } S))(o)$ into $(\mathrm{FreeSorts}(X) \cdot (\text{the result sort of } S))(o)$ is defined by:

(Def.14) For every finite sequence p of elements of $\mathrm{TS}(\mathrm{DTConMSA}(X))$ such that $\mathrm{Sym}(o, X) \Rightarrow$ the roots of p holds $(\mathrm{DenOp}(o, X))(p) = \mathrm{Sym}(o, X)\text{-tree}(p)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. The functor $\mathrm{FreeOperations}(X)$ yielding a many sorted function from $(\mathrm{FreeSorts}(X))^{\#} \cdot (\text{the arity of } S)$ into $\mathrm{FreeSorts}(X) \cdot (\text{the result sort of } S)$ is defined as follows:

(Def.15) For every operation symbol o of S holds $(\mathrm{FreeOperations}(X))(o) = \mathrm{DenOp}(o, X)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. The functor $\mathrm{Free}(X)$ yields a strict non-empty algebra over S and is defined by:

(Def.16) $\mathrm{Free}(X) = \langle \mathrm{FreeSorts}(X), \mathrm{FreeOperations}(X) \rangle$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, and let s be a sort symbol of S. The functor $\mathrm{FreeGenerator}(s, X)$ yields a non empty subset of $(\mathrm{FreeSorts}(X))(s)$ and is defined as follows:

(Def.17) For arbitrary x holds $x \in \mathrm{FreeGenerator}(s, X)$ iff there exists arbitrary a such that $a \in X(s)$ and $x =$ the root tree of $\langle a, s \rangle$.

The following proposition is true

(14) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let s be a sort symbol of S. Then $\mathrm{FreeGenerator}(s, X) = \{$the root tree of t: t ranges over symbols of $\mathrm{DTConMSA}(X)$, $t \in$ the terminals of $\mathrm{DTConMSA}(X) \ \wedge \ t_{\mathbf{2}} = s\}$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. The functor $\mathrm{FreeGenerator}(X)$ yielding a generator set of $\mathrm{Free}(X)$ is defined as follows:

(Def.18) For every sort symbol s of S holds $(\mathrm{FreeGenerator}(X))(s) = \mathrm{FreeGenerator}(s, X)$.

We now state two propositions:

(15) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Then $\mathrm{FreeGenerator}(X)$ is non-empty.

(16) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Then

$\bigcup$rng FreeGenerator(X) = {the root tree of t: t ranges over symbols of DTConMSA(X), $t \in$ the terminals of DTConMSA(X)}.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, and let s be a sort symbol of S. The functor Reverse(s, X) yielding a function from FreeGenerator(s, X) into $X(s)$ is defined as follows:

(Def.19) For every symbol t of DTConMSA(X) such that the root tree of $t \in$ FreeGenerator(s, X) holds (Reverse(s, X))(the root tree of t) $= t_{\mathbf{1}}$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. The functor Reverse(X) yielding a many sorted function from FreeGenerator(X) into X is defined by:

(Def.20) For every sort symbol s of S holds (Reverse(X))(s) = Reverse(s, X).

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, let A be a non-empty many sorted set of the carrier of S, let F be a many sorted function from FreeGenerator(X) into A, and let t be a symbol of DTConMSA(X). Let us assume that $t \in$ the terminals of DTConMSA(X). The functor $\pi(F, A, t)$ yielding an element of $\bigcup A$ is defined as follows:

(Def.21) For every function f such that $f = F(t_{\mathbf{2}})$ holds $\pi(F, A, t) = f$(the root tree of t).

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, and let t be a symbol of DTConMSA(X). Let us assume that there exists a finite sequence p such that $t \Rightarrow p$. The functor ${}^{@}(X, t)$ yielding an operation symbol of S is defined by:

(Def.22) $\langle {}^{@}(X, t)$, the carrier of $S \rangle = t$.

Let S be a non void non empty many sorted signature, let U_0 be a non-empty algebra over S, let o be an operation symbol of S, and let p be a finite sequence. Let us assume that $p \in$ Args(o, U_0). The functor $\pi(o, U_0, p)$ yielding an element of $\bigcup$(the sorts of U_0) is defined by:

(Def.23) $\pi(o, U_0, p) = (\mathrm{Den}(o, U_0))(p)$.

Next we state two propositions:

(17) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Then FreeGenerator(X) is free.

(18) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Then Free(X) is free.

Let S be a non void non empty many sorted signature. One can check that there exists a non-empty algebra over S which is free and strict.

Let S be a non void non empty many sorted signature and let U_0 be a free non-empty algebra over S. One can verify that there exists a generator set of U_0 which is free.

One can prove the following propositions:

(19) Let S be a non void non empty many sorted signature and let U_1 be a non-empty algebra over S. Then there exists a strict free non-empty algebra U_0 over S such that there exists many sorted function from U_0 into U_1 which is an epimorphism of U_0 onto U_1.

(20) Let S be a non void non empty many sorted signature and let U_1 be a strict non-empty algebra over S. Then there exists a strict free non-empty algebra U_0 over S and there exists a many sorted function F from U_0 into U_1 such that F is an epimorphism of U_0 onto U_1 and $\operatorname{Im} F = U_1$.

References

[1] Grzegorz Bancerek. Cartesian product of functions. *Formalized Mathematics*, 2(**4**):547–552, 1991.

[2] Grzegorz Bancerek. Introduction to trees. *Formalized Mathematics*, 1(**2**):421–427, 1990.

[3] Grzegorz Bancerek. Joining of decorated trees. *Formalized Mathematics*, 4(**1**):77–82, 1993.

[4] Grzegorz Bancerek. König's lemma. *Formalized Mathematics*, 2(**3**):397–402, 1991.

[5] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[6] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Formalized Mathematics*, 3(**2**):195–204, 1992.

[7] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[8] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Formalized Mathematics*, 4(**1**):91–101, 1993.

[9] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Formalized Mathematics*, 5(**1**):47–54, 1996.

[10] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[11] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[12] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[13] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[14] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar - part 1. *Formalized Mathematics*, 2(**5**):683–687, 1991.

[15] Mał gorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Formalized Mathematics*, 5(**1**):61–65, 1996.

[16] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Formalized Mathematics*, 3(**2**):251–253, 1992.

[17] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.

[18] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[19] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.

[20] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.

[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[22] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[23] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[25] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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