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On the Decomposition of the States of SCM

Yasushi Tanaka
Shinshu University
Information Engineering Dept.
Nagano

Summary. This article continues the development of the basic terminology for the **SCM** as defined in [11,12,18]. There is developed of the terminology for discussing static properties of instructions (i.e. not related to execution), for data locations, instruction locations, as well as for states and partial states of **SCM**. The main contribution of the article consists in characterizing **SCM** computations starting in states containing autonomic finite partial states.

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The articles [17], [2], [16], [10], [15], [20], [5], [6], [7], [19], [1], [14], [4], [9], [3], [8], [11], [12], [18], and [13] provide the notation and terminology for this paper.

1. Preliminaries

The following propositions are true:

(1) For all sets A, B, X, Y such that $A \subseteq X$ and $B \subseteq Y$ and $X \cap Y = \emptyset$ holds $A \cap B = \emptyset$.

(2) For all sets X, Y, Z such that $X \subseteq Y$ holds $X \subseteq Z \cup Y$ and $X \subseteq Y \cup Z$.

(3) For all natural numbers m, k such that $k \neq 0$ holds $m \cdot k \div k = m$.

(4) For all natural numbers i, j such that $i \geq j$ holds $i -' j + j = i$.

(5) For all functions f, g and for all sets A, B such that $A \subseteq B$ and $f \restriction B = g \restriction B$ holds $f \restriction A = g \restriction A$.

(6) For all functions p, q and for every set A holds $(p+\cdot q) \restriction A = p \restriction A +\cdot q \restriction A$.

(7) For all functions f, g and for every set A such that A misses $\mathrm{dom}\, g$ holds $(f +\cdot g) \restriction A = f \restriction A$.

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(8) For all functions f, g and for every set A such that $\operatorname{dom} f$ misses A holds $(f +\cdot g) \restriction A = g \restriction A$.

(9) For all functions f, g, h such that $\operatorname{dom} g = \operatorname{dom} h$ holds $f +\cdot g +\cdot h = f +\cdot h$.

(10) For all functions f, g such that $f \subseteq g$ holds $f +\cdot g = g$ and $g +\cdot f = g$.

(11) For every function f and for every set A holds $f +\cdot f \restriction A = f$.

(12) For all functions f, g and for all sets B, C such that $\operatorname{dom} f \subseteq B$ and $\operatorname{dom} g \subseteq C$ and B misses C holds $(f +\cdot g) \restriction B = f$ and $(f +\cdot g) \restriction C = g$.

(13) For all functions p, q and for every set A such that $\operatorname{dom} p \subseteq A$ and $\operatorname{dom} q$ misses A holds $(p +\cdot q) \restriction A = p$.

(14) For every function f and for all sets A, B holds $f \restriction (A \cup B) = f \restriction A +\cdot f \restriction B$.

2. Total states of **SCM**

One can prove the following propositions:

(15) Let a be a data-location and let s be a state of **SCM**. Then $(\mathrm{Exec}(\mathrm{Divide}(a, a), s))(\mathbf{IC}_{\mathbf{SCM}}) = \mathrm{Next}(\mathbf{IC}_s)$ and $(\mathrm{Exec}(\mathrm{Divide}(a, a), s))(a) = s(a) \bmod s(a)$ and for every data-location c such that $c \neq a$ holds $(\mathrm{Exec}(\mathrm{Divide}(a, a), s))(c) = s(c)$.

(16) For arbitrary x such that $x \in \text{Data-Loc}_{\text{SCM}}$ holds x is a data-location.

(17) For arbitrary x such that $x \in \text{Instr-Loc}_{\text{SCM}}$ holds x is an instruction-location of **SCM**.

(18) For every data-location d_1 there exists a natural number i such that $d_1 = \mathbf{d}_i$.

(19) For every instruction-location i_1 of **SCM** there exists a natural number i such that $i_1 = \mathbf{i}_i$.

(20) For every data-location d_1 holds $d_1 \neq \mathbf{IC}_{\mathbf{SCM}}$.

(21) For every instruction-location i_1 of **SCM** holds $i_1 \neq \mathbf{IC}_{\mathbf{SCM}}$.

(22) For every instruction-location i_1 of **SCM** and for every data-location d_1 holds $i_1 \neq d_1$.

(23) The objects of $\mathbf{SCM} = \{\mathbf{IC}_{\mathbf{SCM}}\} \cup \text{Data-Loc}_{\text{SCM}} \cup \text{Instr-Loc}_{\text{SCM}}$.

(24) Let s be a state of **SCM**, and let d be a data-location, and let l be an instruction-location of **SCM**. Then $d \in \operatorname{dom} s$ and $l \in \operatorname{dom} s$.

(25) For every state s of **SCM** holds $\mathbf{IC}_{\mathbf{SCM}} \in \operatorname{dom} s$.

(26) Let s_1, s_2 be states of **SCM**. Suppose $\mathbf{IC}_{(s_1)} = \mathbf{IC}_{(s_2)}$ and for every data-location a holds $s_1(a) = s_2(a)$ and for every instruction-location i of **SCM** holds $s_1(i) = s_2(i)$. Then $s_1 = s_2$.

(27) For every state s of **SCM** holds $\text{Data-Loc}_{\text{SCM}} \subseteq \operatorname{dom} s$.

(28) For every state s of **SCM** holds $\text{Instr-Loc}_{\text{SCM}} \subseteq \operatorname{dom} s$.

(29) For every state s of **SCM** holds $\operatorname{dom}(s \restriction \text{Data-Loc}_{\text{SCM}}) = \text{Data-Loc}_{\text{SCM}}$.

(30) For every state s of **SCM** holds $\mathrm{dom}(s \restriction \mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}) = \mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$.

(31) $\mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$ is finite.

(32) The instruction locations of **SCM** is finite.

(33) $\mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$ misses $\mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$.

(34) For every state s of **SCM** holds $\mathrm{Start\text{-}At}(\mathbf{IC}_s) = s \restriction \{\mathbf{IC}_{\mathbf{SCM}}\}$.

(35) For every instruction-location l of **SCM** holds $\mathrm{Start\text{-}At}(l) = \{\langle \mathbf{IC}_{\mathbf{SCM}}, l \rangle\}$.

Let I be an instruction of **SCM**. The functor $\mathrm{InsCode}(I)$ yields a natural number and is defined as follows:

(Def.1) $\mathrm{InsCode}(I) = I_{\mathbf{1}}$.

The functor ${}^{@}I$ yielding an element of $\mathrm{Instr}_{\mathrm{SCM}}$ is defined by:

(Def.2) ${}^{@}I = I$.

Let l_1 be an element of $\mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$. The functor ${l_1}^{\mathrm{T}}$ yields an instruction-location of **SCM** and is defined as follows:

(Def.3) ${l_1}^{\mathrm{T}} = l_1$.

Let l_1 be an element of $\mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$. The functor ${l_1}^{\mathrm{T}}$ yielding a data-location is defined as follows:

(Def.4) ${l_1}^{\mathrm{T}} = l_1$.

One can prove the following proposition

(36) For every instruction l of **SCM** holds $\mathrm{InsCode}(l) \leq 8$.

In the sequel a, b are data-locations and l_1 is an instruction-location of **SCM**.

One can prove the following propositions:

(37) $\mathrm{InsCode}(\mathbf{halt}_{\mathbf{SCM}}) = 0$.

(38) $\mathrm{InsCode}(a{:=}b) = 1$.

(39) $\mathrm{InsCode}(\mathrm{AddTo}(a, b)) = 2$.

(40) $\mathrm{InsCode}(\mathrm{SubFrom}(a, b)) = 3$.

(41) $\mathrm{InsCode}(\mathrm{MultBy}(a, b)) = 4$.

(42) $\mathrm{InsCode}(\mathrm{Divide}(a, b)) = 5$.

(43) $\mathrm{InsCode}(\mathrm{goto}\ l_1) = 6$.

(44) $\mathrm{InsCode}(\mathbf{if}\ a = 0\ \mathbf{goto}\ l_1) = 7$.

(45) $\mathrm{InsCode}(\mathbf{if}\ a > 0\ \mathbf{goto}\ l_1) = 8$.

In the sequel d_2, d_3 denote data-locations and l_1 denotes an instruction-location of **SCM**.

We now state a number of propositions:

(46) For every instruction i_2 of **SCM** such that $\mathrm{InsCode}(i_2) = 0$ holds $i_2 = \mathbf{halt}_{\mathbf{SCM}}$.

(47) For every instruction i_2 of **SCM** such that $\mathrm{InsCode}(i_2) = 1$ there exist d_2, d_3 such that $i_2 = d_2{:=}d_3$.

(48) For every instruction i_2 of **SCM** such that $\mathrm{InsCode}(i_2) = 2$ there exist d_2, d_3 such that $i_2 = \mathrm{AddTo}(d_2, d_3)$.

(49) For every instruction i_2 of **SCM** such that $\mathrm{InsCode}(i_2) = 3$ there exist d_2, d_3 such that $i_2 = \mathrm{SubFrom}(d_2, d_3)$.

(50) For every instruction i_2 of **SCM** such that $\mathrm{InsCode}(i_2) = 4$ there exist d_2, d_3 such that $i_2 = \mathrm{MultBy}(d_2, d_3)$.

(51) For every instruction i_2 of **SCM** such that $\mathrm{InsCode}(i_2) = 5$ there exist d_2, d_3 such that $i_2 = \mathrm{Divide}(d_2, d_3)$.

(52) For every instruction i_2 of **SCM** such that $\mathrm{InsCode}(i_2) = 6$ there exists l_1 such that $i_2 = \mathrm{goto}\ l_1$.

(53) For every instruction i_2 of **SCM** such that $\mathrm{InsCode}(i_2) = 7$ there exist l_1, d_2 such that $i_2 = \mathbf{if}\ d_2 = 0\ \mathbf{goto}\ l_1$.

(54) For every instruction i_2 of **SCM** such that $\mathrm{InsCode}(i_2) = 8$ there exist l_1, d_2 such that $i_2 = \mathbf{if}\ d_2 > 0\ \mathbf{goto}\ l_1$.

(55) For every instruction-location l_1 of **SCM** holds $({}^{@}\mathrm{goto}\ l_1)\mathrm{address}_{\mathrm{j}} = l_1$.

(56) For every instruction-location l_1 of **SCM** and for every data-location a holds $({}^{@}(\mathbf{if}\ a = 0\ \mathbf{goto}\ l_1))\mathrm{address}_{\mathrm{j}} = l_1$ and $({}^{@}(\mathbf{if}\ a = 0\ \mathbf{goto}\ l_1))\mathrm{address}_{\mathrm{c}} = a$.

(57) For every instruction-location l_1 of **SCM** and for every data-location a holds $({}^{@}(\mathbf{if}\ a > 0\ \mathbf{goto}\ l_1))\mathrm{address}_{\mathrm{j}} = l_1$ and $({}^{@}(\mathbf{if}\ a > 0\ \mathbf{goto}\ l_1))\mathrm{address}_{\mathrm{c}} = a$.

(58) For all states s_1, s_2 of **SCM** such that $s_1 \restriction (\mathrm{Data\text{-}Loc}_{\mathrm{SCM}} \cup \{\mathbf{IC}_{\mathbf{SCM}}\}) = s_2 \restriction (\mathrm{Data\text{-}Loc}_{\mathrm{SCM}} \cup \{\mathbf{IC}_{\mathbf{SCM}}\})$ and for every instruction l of **SCM** holds $\mathrm{Exec}(l, s_1) \restriction (\mathrm{Data\text{-}Loc}_{\mathrm{SCM}} \cup \{\mathbf{IC}_{\mathbf{SCM}}\}) = \mathrm{Exec}(l, s_2) \restriction (\mathrm{Data\text{-}Loc}_{\mathrm{SCM}} \cup \{\mathbf{IC}_{\mathbf{SCM}}\})$.

(59) For every instruction i of **SCM** and for every state s of **SCM** holds $\mathrm{Exec}(i, s) \restriction \mathrm{Instr\text{-}Loc}_{\mathrm{SCM}} = s \restriction \mathrm{Instr\text{-}Loc}_{\mathrm{SCM}}$.

3. Finite partial states of **SCM**

The following proposition is true

(60) For every finite partial state p of **SCM** and for every state s of **SCM** such that $\mathbf{IC}_{\mathbf{SCM}} \in \mathrm{dom}\, p$ and $p \subseteq s$ holds $\mathbf{IC}_p = \mathbf{IC}_s$.

Let p be a finite partial state of **SCM** and let l_1 be an instruction-location of **SCM**. Let us assume that $l_1 \in \mathrm{dom}\, p$. The functor $\pi_{l_1} p$ yielding an instruction of **SCM** is defined by:

(Def.5) $\pi_{l_1} p = p(l_1)$.

The following proposition is true

(61) Let x be arbitrary and let p be a finite partial state of **SCM**. If $x \subseteq p$, then x is a finite partial state of **SCM**.

Let p be a finite partial state of **SCM**. The functor $\mathrm{ProgramPart}(p)$ yields a programmed finite partial state of **SCM** and is defined by:

(Def.6) $\mathrm{ProgramPart}(p) = p \restriction (\text{the instruction locations of } \mathbf{SCM})$.

The functor DataPart(p) yielding a finite partial state of **SCM** is defined as follows:

(Def.7) DataPart(p) = $p \upharpoonright$ Data-Loc$_{\rm SCM}$.

A finite partial state of **SCM** is data-only if:

(Def.8) dom it $\subseteq$ Data-Loc$_{\rm SCM}$.

Next we state a number of propositions:

(62) For every finite partial state p of **SCM** holds DataPart(p) $\subseteq p$.

(63) For every finite partial state p of **SCM** holds ProgramPart(p) $\subseteq p$.

(64) Let p be a finite partial state of **SCM** and let s be a state of **SCM**. If $p \subseteq s$, then for every natural number i holds ProgramPart(p) $\subseteq$ (Computation(s))(i).

(65) For every finite partial state p of **SCM** holds $\mathbf{IC}_{\mathbf{SCM}} \notin$ dom DataPart(p).

(66) For every finite partial state p of **SCM** holds $\mathbf{IC}_{\mathbf{SCM}} \notin$ dom ProgramPart(p).

(67) For every finite partial state p of **SCM** holds $\{\mathbf{IC}_{\mathbf{SCM}}\}$ misses dom DataPart(p).

(68) For every finite partial state p of **SCM** holds $\{\mathbf{IC}_{\mathbf{SCM}}\}$ misses dom ProgramPart(p).

(69) For every finite partial state p of **SCM** holds dom DataPart(p) $\subseteq$ Data-Loc$_{\rm SCM}$.

(70) For every finite partial state p of **SCM** holds dom ProgramPart(p) $\subseteq$ Instr-Loc$_{\rm SCM}$.

(71) For all finite partial states p, q of **SCM** holds dom DataPart(p) misses dom ProgramPart(q).

(72) For every programmed finite partial state p of **SCM** holds ProgramPart(p) = p.

(73) For every finite partial state p of **SCM** and for every instruction-location l of **SCM** such that $l \in$ dom p holds $l \in$ dom ProgramPart(p).

(74) Let p be a data-only finite partial state of **SCM** and let q be a finite partial state of **SCM**. Then $p \subseteq q$ if and only if $p \subseteq$ DataPart(q).

(75) For every finite partial state p of **SCM** such that $\mathbf{IC}_{\mathbf{SCM}} \in$ dom p holds p = Start-At($\mathbf{IC}_p$) +· ProgramPart(p) +· DataPart(p).

A partial function from FinPartSt(**SCM**) to FinPartSt(**SCM**) is data-only if it satisfies the condition (Def.9).

(Def.9) Let p be a finite partial state of **SCM**. Suppose $p \in$ dom it. Then p is data-only and for every finite partial state q of **SCM** such that q = it(p) holds q is data-only.

Next we state the proposition

(76) For every finite partial state p of **SCM** such that $\mathbf{IC}_{\mathbf{SCM}} \in$ dom p holds p is not programmed.

Let s be a state of **SCM** and let p be a finite partial state of **SCM**. Then $s +\cdot p$ is a state of **SCM**.

Next we state several propositions:

(77) Let i be an instruction of **SCM**, and let s be a state of **SCM**, and let p be a programmed finite partial state of **SCM**. Then $\mathrm{Exec}(i, s +\cdot p) = \mathrm{Exec}(i, s) +\cdot p$.

(78) For every finite partial state p of **SCM** such that $\mathbf{IC}_{\mathbf{SCM}} \in \mathrm{dom}\, p$ holds $\text{Start-At}(\mathbf{IC}_p) \subseteq p$.

(79) For every state s of **SCM** and for every instruction-location i_3 of **SCM** holds $\mathbf{IC}_{s +\cdot \text{Start-At}(i_3)} = i_3$.

(80) For every state s of **SCM** and for every instruction-location i_3 of **SCM** and for every data-location a holds $s(a) = (s +\cdot \text{Start-At}(i_3))(a)$.

(81) Let s be a state of **SCM**, and let i_3 be an instruction-location of **SCM**, and let a be an instruction-location of **SCM**. Then $s(a) = (s +\cdot \text{Start-At}(i_3))(a)$.

(82) For all states s, t of **SCM** holds $s +\cdot t \restriction \text{Data-Loc}_{\mathrm{SCM}}$ is a state of **SCM**.

4. Autonomic finite partial states of SCM

The following proposition is true

(83) For every autonomic finite partial state p of **SCM** such that $\mathrm{DataPart}(p) \neq \emptyset$ holds $\mathbf{IC}_{\mathbf{SCM}} \in \mathrm{dom}\, p$.

One can check that there exists a finite partial state of **SCM** which is autonomic and non programmed.

We now state a number of propositions:

(84) For every autonomic non programmed finite partial state p of **SCM** holds $\mathbf{IC}_{\mathbf{SCM}} \in \mathrm{dom}\, p$.

(85) For every autonomic finite partial state p of **SCM** such that $\mathbf{IC}_{\mathbf{SCM}} \in \mathrm{dom}\, p$ holds $\mathbf{IC}_p \in \mathrm{dom}\, p$.

(86) Let p be an autonomic non programmed finite partial state of **SCM** and let s be a state of **SCM**. If $p \subseteq s$, then for every natural number i holds $\mathbf{IC}_{(\mathrm{Computation}(s))(i)} \in \mathrm{dom}\, \mathrm{ProgramPart}(p)$.

(87) Let p be an autonomic non programmed finite partial state of **SCM** and let s_1, s_2 be states of **SCM**. Suppose $p \subseteq s_1$ and $p \subseteq s_2$. Let i be a natural number, and let d_2, d_3 be data-locations, and let l_1 be an instruction-location of **SCM**, and let I be an instruction of **SCM**. If $I = \mathrm{CurInstr}((\mathrm{Computation}(s_1))(i))$, then $\mathbf{IC}_{(\mathrm{Computation}(s_1))(i)} = \mathbf{IC}_{(\mathrm{Computation}(s_2))(i)}$ and $I = \mathrm{CurInstr}((\mathrm{Computation}(s_2))(i))$.

(88) Let p be an autonomic non programmed finite partial state of **SCM** and let s_1, s_2 be states of **SCM**. Suppose $p \subseteq s_1$ and $p \subseteq s_2$. Let i be a natural number, and let d_2, d_3 be data-locations, and let l_1 be an

instruction-location of **SCM**, and let I be an instruction of **SCM**. If $I = \mathrm{CurInstr}((\mathrm{Computation}(s_1))(i))$, then if $I = d_2{:=}d_3$ and $d_2 \in \mathrm{dom}\, p$, then $(\mathrm{Computation}(s_1))(i)(d_3) = (\mathrm{Computation}(s_2))(i)(d_3)$.

(89) Let p be an autonomic non programmed finite partial state of **SCM** and let s_1, s_2 be states of **SCM**. Suppose $p \subseteq s_1$ and $p \subseteq s_2$. Let i be a natural number, and let d_2, d_3 be data-locations, and let l_1 be an instruction-location of **SCM**, and let I be an instruction of **SCM**. Suppose $I = \mathrm{CurInstr}((\mathrm{Computation}(s_1))(i))$. If $I = \mathrm{AddTo}(d_2, d_3)$ and $d_2 \in \mathrm{dom}\, p$, then $(\mathrm{Computation}(s_1))(i)(d_2) + (\mathrm{Computation}(s_1))(i)(d_3) = (\mathrm{Computation}(s_2))(i)(d_2) + (\mathrm{Computation}(s_2))(i)(d_3)$.

(90) Let p be an autonomic non programmed finite partial state of **SCM** and let s_1, s_2 be states of **SCM**. Suppose $p \subseteq s_1$ and $p \subseteq s_2$. Let i be a natural number, and let d_2, d_3 be data-locations, and let l_1 be an instruction-location of **SCM**, and let I be an instruction of **SCM**. Suppose $I = \mathrm{CurInstr}((\mathrm{Computation}(s_1))(i))$. If $I = \mathrm{SubFrom}(d_2, d_3)$ and $d_2 \in \mathrm{dom}\, p$, then $(\mathrm{Computation}(s_1))(i)(d_2) - (\mathrm{Computation}(s_1))(i)(d_3) = (\mathrm{Computation}(s_2))(i)(d_2) - (\mathrm{Computation}(s_2))(i)(d_3)$.

(91) Let p be an autonomic non programmed finite partial state of **SCM** and let s_1, s_2 be states of **SCM**. Suppose $p \subseteq s_1$ and $p \subseteq s_2$. Let i be a natural number, and let d_2, d_3 be data-locations, and let l_1 be an instruction-location of **SCM**, and let I be an instruction of **SCM**. Suppose $I = \mathrm{CurInstr}((\mathrm{Computation}(s_1))(i))$. If $I = \mathrm{MultBy}(d_2, d_3)$ and $d_2 \in \mathrm{dom}\, p$, then $(\mathrm{Computation}(s_1))(i)(d_2) \cdot (\mathrm{Computation}(s_1))(i)(d_3) = (\mathrm{Computation}(s_2))(i)(d_2) \cdot (\mathrm{Computation}(s_2))(i)(d_3)$.

(92) Let p be an autonomic non programmed finite partial state of **SCM** and let s_1, s_2 be states of **SCM**. Suppose $p \subseteq s_1$ and $p \subseteq s_2$. Let i be a natural number, and let d_2, d_3 be data-locations, and let l_1 be an instruction-location of **SCM**, and let I be an instruction of **SCM**. Suppose $I = \mathrm{CurInstr}((\mathrm{Computation}(s_1))(i))$. If $I = \mathrm{Divide}(d_2, d_3)$ and $d_2 \in \mathrm{dom}\, p$ and $d_2 \neq d_3$, then $(\mathrm{Computation}(s_1))(i)(d_2) \div (\mathrm{Computation}(s_1))(i)(d_3) = (\mathrm{Computation}(s_2))(i)(d_2) \div (\mathrm{Computation}(s_2))(i)(d_3)$.

(93) Let p be an autonomic non programmed finite partial state of **SCM** and let s_1, s_2 be states of **SCM**. Suppose $p \subseteq s_1$ and $p \subseteq s_2$. Let i be a natural number, and let d_2, d_3 be data-locations, and let l_1 be an instruction-location of **SCM**, and let I be an instruction of **SCM**. Suppose $I = \mathrm{CurInstr}((\mathrm{Computation}(s_1))(i))$. If $I = \mathrm{Divide}(d_2, d_3)$ and $d_3 \in \mathrm{dom}\, p$ and $d_2 \neq d_3$, then $(\mathrm{Computation}(s_1))(i)(d_2) \mathrm{~mod~} (\mathrm{Computation}(s_1))(i)(d_3) = (\mathrm{Computation}(s_2))(i)(d_2) \mathrm{~mod~} (\mathrm{Computation}(s_2))(i)(d_3)$.

(94) Let p be an autonomic non programmed finite partial state of **SCM** and let s_1, s_2 be states of **SCM**. Suppose $p \subseteq s_1$ and $p \subseteq s_2$. Let i be a natural number, and let d_2, d_3 be data-locations, and let l_1 be an instruction-location of **SCM**, and let I be an instruction of **SCM**. Suppose $I = \mathrm{CurInstr}((\mathrm{Computation}(s_1))(i))$. If $I = \textbf{if } d_2 = 0 \textbf{ goto } l_1$ and $l_1 \neq \mathrm{Next}(\mathbf{IC}_{(\mathrm{Computation}(s_1))(i)})$, then $(\mathrm{Computation}(s_1))(i)(d_2) = 0$

iff $(\mathrm{Computation}(s_2))(i)(d_2) = 0$.

(95) Let p be an autonomic non programmed finite partial state of **SCM** and let s_1, s_2 be states of **SCM**. Suppose $p \subseteq s_1$ and $p \subseteq s_2$. Let i be a natural number, and let d_2, d_3 be data-locations, and let l_1 be an instruction-location of **SCM**, and let I be an instruction of **SCM**. Suppose $I = \mathrm{CurInstr}((\mathrm{Computation}(s_1))(i))$. If $I =$ **if** $d_2 > 0$ **goto** l_1 and $l_1 \neq \mathrm{Next}(\mathbf{IC}_{(\mathrm{Computation}(s_1))(i)})$, then $(\mathrm{Computation}(s_1))(i)(d_2) > 0$ iff $(\mathrm{Computation}(s_2))(i)(d_2) > 0$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[8] Czesław Byliński. Products and coproducts in categories. *Formalized Mathematics*, 2(**5**):701–709, 1991.
[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[11] Yatsuka Nakamura and Andrzej Trybulec. A mathematical model of CPU. *Formalized Mathematics*, 3(**2**):151–160, 1992.
[12] Yatsuka Nakamura and Andrzej Trybulec. On a mathematical model of programs. *Formalized Mathematics*, 3(**2**):241–250, 1992.
[13] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.
[14] Dariusz Surowik. Cyclic groups and some of their properties - part I. *Formalized Mathematics*, 2(**5**):623–627, 1991.
[15] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[16] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[18] Andrzej Trybulec and Yatsuka Nakamura. Some remarks on the simple concrete model of computer. *Formalized Mathematics*, 4(**1**):51–56, 1993.
[19] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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On Defining Functions on Binary Trees [1]

Grzegorz Bancerek
Polish Academy of Sciences
Institute of Mathematics
Warsaw

Piotr Rudnicki
University of Alberta
Department of Computing Science
Edmonton

Summary. This article is a continuation of an article on defining functions on trees (see [6]). In this article we develop terminology specialized for binary trees, first defining binary trees and binary grammars. We recast the induction principle for the set of parse trees of binary grammars and the scheme of defining functions inductively with the set as domain. We conclude with defining the scheme of defining such functions by lambda-like expressions.

MML Identifier: `BINTREE1`.

The terminology and notation used here are introduced in the following articles: [12], [14], [15], [13], [8], [9], [5], [7], [11], [10], [1], [3], [4], [2], and [6].

Let D be a non empty set and let t be a tree decorated with elements of D. The root label of t is an element of D and is defined by:

(Def.1) The root label of $t = t(\varepsilon)$.

One can prove the following two propositions:

(1) Let D be a non empty set and let t be a tree decorated with elements of D. Then the roots of $\langle t \rangle = \langle$the root label of $t\rangle$.

(2) Let D be a non empty set and let t_1, t_2 be trees decorated with elements of D. Then the roots of $\langle t_1, t_2 \rangle = \langle$the root label of t_1, the root label of $t_2\rangle$.

A tree is binary if:

(Def.2) For every element t of it such that $t \notin$ Leaves(it) holds $\mathrm{succ}\, t = \{t \frown \langle 0 \rangle, t \frown \langle 1 \rangle\}$.

The following propositions are true:

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(3) For every tree T and for every element t of T holds $t \in \mathrm{Leaves}(T)$ iff $t \frown \langle 0 \rangle \notin T$.

(4) For every tree T and for every element t of T holds $t \in \mathrm{Leaves}(T)$ iff it is not true that there exists a natural number n such that $t \frown \langle n \rangle \in T$.

(5) For every tree T and for every element t of T holds $\mathrm{succ}\, t = \emptyset$ iff $t \in \mathrm{Leaves}(T)$.

(6) The elementary tree of 0 is binary.

(7) The elementary tree of 2 is binary.

Let us note that there exists a tree which is binary and finite.

A decorated tree is binary if:

(Def.3) dom it is binary.

Let D be a non empty set. Observe that there exists a tree decorated with elements of D which is binary and finite.

Let us mention that there exists a decorated tree which is binary and finite.

Let us observe that every tree which is binary is also finite-order.

We now state four propositions:

(8) Let T_0, T_1 be trees and let t be an element of $\overbrace{T_0, T_1}$. Then

(i) for every element p of T_0 such that $t = \langle 0 \rangle \frown p$ holds $t \in \mathrm{Leaves}(\overbrace{T_0, T_1})$ iff $p \in \mathrm{Leaves}(T_0)$, and

(ii) for every element p of T_1 such that $t = \langle 1 \rangle \frown p$ holds $t \in \mathrm{Leaves}(\overbrace{T_0, T_1})$ iff $p \in \mathrm{Leaves}(T_1)$.

(9) Let T_0, T_1 be trees and let t be an element of $\overbrace{T_0, T_1}$. Then

(i) if $t = \varepsilon$, then $\mathrm{succ}\, t = \{t \frown \langle 0 \rangle, t \frown \langle 1 \rangle\}$,

(ii) for every element p of T_0 such that $t = \langle 0 \rangle \frown p$ and for every finite sequence s_1 holds $s_1 \in \mathrm{succ}\, p$ iff $\langle 0 \rangle \frown s_1 \in \mathrm{succ}\, t$, and

(iii) for every element p of T_1 such that $t = \langle 1 \rangle \frown p$ and for every finite sequence s_1 holds $s_1 \in \mathrm{succ}\, p$ iff $\langle 1 \rangle \frown s_1 \in \mathrm{succ}\, t$.

(10) For all trees T_1, T_2 holds T_1 is binary and T_2 is binary iff $\overbrace{T_1, T_2}$ is binary.

(11) For all decorated trees T_1, T_2 and for arbitrary x holds T_1 is binary and T_2 is binary iff x-tree(T_1, T_2) is binary.

Let D be a non empty set, let x be an element of D, and let T_1, T_2 be binary finite trees decorated with elements of D. Then x-tree(T_1, T_2) is a binary finite tree decorated with elements of D.

A non empty tree construction structure is binary if:

(Def.4) For every symbol s of it and for every finite sequence p such that $s \Rightarrow p$ there exist symbols x_1, x_2 of it such that $p = \langle x_1, x_2 \rangle$.

One can check that there exists a non empty tree construction structure which is binary and strict and has terminals, nonterminals, and useful nonterminals.

The scheme *BinDTConstrStrEx* concerns a non empty set $\mathcal{A}$ and a ternary predicate $\mathcal{P}$, and states that:

There exists a binary strict non empty tree construction structure G such that the carrier of $G = \mathcal{A}$ and for all symbols x, y, z of G holds $x \Rightarrow \langle y, z \rangle$ iff $\mathcal{P}[x, y, z]$

for all values of the parameters.

One can prove the following proposition

(12) Let G be a binary non empty tree construction structure with terminals and nonterminals, and let t_3 be a finite sequence of elements of $\mathrm{TS}(G)$, and let n_1 be a symbol of G. Suppose $n_1 \Rightarrow$ the roots of t_3. Then

(i) n_1 is a nonterminal of G,

(ii) $\mathrm{dom}\, t_3 = \{1, 2\}$,

(iii) $1 \in \mathrm{dom}\, t_3$,

(iv) $2 \in \mathrm{dom}\, t_3$, and

(v) there exist elements t_4, t_5 of $\mathrm{TS}(G)$ such that the roots of $t_3 = \langle$the root label of t_4, the root label of $t_5\rangle$ and $t_4 = t_3(1)$ and $t_5 = t_3(2)$ and n_1-tree$(t_3) = n_1$-tree(t_4, t_5) and $t_4 \in \mathrm{rng}\, t_3$ and $t_5 \in \mathrm{rng}\, t_3$.

Now we present three schemes. The scheme *BinDTConstrInd* concerns a binary non empty tree construction structure $\mathcal{A}$ with terminals and nonterminals and a unary predicate $\mathcal{P}$, and states that:

For every element t of $\mathrm{TS}(\mathcal{A})$ holds $\mathcal{P}[t]$

provided the parameters have the following properties:

- For every terminal s of $\mathcal{A}$ holds $\mathcal{P}[$the root tree of $s]$,
- Let n_1 be a nonterminal of $\mathcal{A}$ and let t_4, t_5 be elements of $\mathrm{TS}(\mathcal{A})$. Suppose $n_1 \Rightarrow \langle$the root label of t_4, the root label of $t_5\rangle$ and $\mathcal{P}[t_4]$ and $\mathcal{P}[t_5]$. Then $\mathcal{P}[n_1\text{-tree}(t_4, t_5)]$.

The scheme *BinDTConstrIndDef* concerns a binary non empty tree construction structure $\mathcal{A}$ with terminals, nonterminals, and useful nonterminals, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a 5-ary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and states that:

There exists a function f from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{B}$ such that

(i) for every terminal t of $\mathcal{A}$ holds $f($the root tree of $t) = \mathcal{F}(t)$, and

(ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_4, t_5 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_4 and $r_2 =$ the root label of t_5 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ and for all elements x_3, x_4 of $\mathcal{B}$ such that $x_3 = f(t_4)$ and $x_4 = f(t_5)$ holds $f(n_1\text{-tree}(t_4, t_5)) = \mathcal{G}(n_1, r_1, r_2, x_3, x_4)$

for all values of the parameters.

The scheme *BinDTConstrUniqDef* deals with a binary non empty tree construction structure $\mathcal{A}$ with terminals, nonterminals, and useful nonterminals, a non empty set $\mathcal{B}$, functions $\mathcal{C}$, $\mathcal{D}$ from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a 5-ary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the following requirements are met:

- (i) For every terminal t of $\mathcal{A}$ holds $\mathcal{C}$(the root tree of t) $= \mathcal{F}(t)$, and
  (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_4, t_5 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_4 and $r_2 =$ the root label of t_5 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ and for all elements x_3, x_4 of $\mathcal{B}$ such that $x_3 = \mathcal{C}(t_4)$ and $x_4 = \mathcal{C}(t_5)$ holds $\mathcal{C}(n_1\text{-tree}(t_4, t_5)) = \mathcal{G}(n_1, r_1, r_2, x_3, x_4)$,
- (i) For every terminal t of $\mathcal{A}$ holds $\mathcal{D}$(the root tree of t) $= \mathcal{F}(t)$, and
  (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_4, t_5 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_4 and $r_2 =$ the root label of t_5 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ and for all elements x_3, x_4 of $\mathcal{B}$ such that $x_3 = \mathcal{D}(t_4)$ and $x_4 = \mathcal{D}(t_5)$ holds $\mathcal{D}(n_1\text{-tree}(t_4, t_5)) = \mathcal{G}(n_1, r_1, r_2, x_3, x_4)$.

Let A, B, C be non empty sets, let a be an element of A, let b be an element of B, and let c be an element of C. Then $\langle a, b, c \rangle$ is an element of $[\![A, B, C]\!]$.

Now we present two schemes. The scheme *BinDTC DefLambda* deals with a binary non empty tree construction structure $\mathcal{A}$ with terminals, nonterminals, and useful nonterminals, non empty sets $\mathcal{B}$, $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a 4-ary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

> There exists a function f from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{C}^{\mathcal{B}}$ such that
> (i) for every terminal t of $\mathcal{A}$ there exists a function g from $\mathcal{B}$ into $\mathcal{C}$ such that $g = f$(the root tree of t) and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{F}(t, a)$, and
> (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_1, t_2 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_1 and $r_2 =$ the root label of t_2 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ there exist functions g, f_1, f_2 from $\mathcal{B}$ into $\mathcal{C}$ such that $g = f(n_1\text{-tree}(t_1, t_2))$ and $f_1 = f(t_1)$ and $f_2 = f(t_2)$ and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{G}(n_1, f_1, f_2, a)$

for all values of the parameters.

The scheme *BinDTC DefLambdaUniq* deals with a binary non empty tree construction structure $\mathcal{A}$ with terminals, nonterminals, and useful nonterminals, non empty sets $\mathcal{B}$, $\mathcal{C}$, functions $\mathcal{D}$, $\mathcal{E}$ from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{C}^{\mathcal{B}}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a 4-ary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

> $\mathcal{D} = \mathcal{E}$

provided the parameters satisfy the following conditions:

- (i) For every terminal t of $\mathcal{A}$ there exists a function g from $\mathcal{B}$ into $\mathcal{C}$ such that $g = \mathcal{D}$(the root tree of t) and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{F}(t, a)$, and
  (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_1, t_2 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_1 and $r_2 =$ the root label of t_2 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ there exist functions g, f_1, f_2 from $\mathcal{B}$ into $\mathcal{C}$ such that $g = \mathcal{D}(n_1\text{-tree}(t_1, t_2))$

and $f_1 = \mathcal{D}(t_1)$ and $f_2 = \mathcal{D}(t_2)$ and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{G}(n_1, f_1, f_2, a)$,

- (i) For every terminal t of $\mathcal{A}$ there exists a function g from $\mathcal{B}$ into $\mathcal{C}$ such that $g = \mathcal{E}$(the root tree of t) and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{F}(t, a)$, and

  (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_1, t_2 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_1 and $r_2 =$ the root label of t_2 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ there exist functions g, f_1, f_2 from $\mathcal{B}$ into $\mathcal{C}$ such that $g = \mathcal{E}(n_1\text{-tree}(t_1, t_2))$ and $f_1 = \mathcal{E}(t_1)$ and $f_2 = \mathcal{E}(t_2)$ and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{G}(n_1, f_1, f_2, a)$.

Let G be a binary non empty tree construction structure with terminals and nonterminals. Note that every element of $\mathrm{TS}(G)$ is binary.

References

[1] Grzegorz Bancerek. Introduction to trees. *Formalized Mathematics*, 1(**2**):421–427, 1990.
[2] Grzegorz Bancerek. Joining of decorated trees. *Formalized Mathematics*, 4(**1**):77–82, 1993.
[3] Grzegorz Bancerek. König's lemma. *Formalized Mathematics*, 2(**3**):397–402, 1991.
[4] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Formalized Mathematics*, 3(**2**):195–204, 1992.
[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[6] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Formalized Mathematics*, 4(**1**):91–101, 1993.
[7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[10] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar - part 1. *Formalized Mathematics*, 2(**5**):683–687, 1991.
[11] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[13] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[14] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[15] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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A Compiler of Arithmetic Expressions for SCM [1]

Grzegorz Bancerek
Polish Academy of Sciences
Institute of Mathematics
Warsaw

Piotr Rudnicki
University of Alberta
Department of Computing Science
Edmonton

Summary. We define a set of binary arithmetic expressions with the following operations: $+$, $-$, $\cdot$, `mod`, and `div` and formalize the common meaning of the expressions in the set of integers. Then, we define a compile function that for a given expression results in a program for the **SCM** machine defined by Nakamura and Trybulec in [13]. We prove that the generated program when loaded into the machine and executed computes the value of the expression. The program uses additional memory and runs in time linear in length of the expression.

MML Identifier: `SCM_COMP`.

The articles [16], [12], [1], [21], [18], [20], [17], [9], [10], [3], [2], [13], [14], [19], [15], [5], [4], [8], [11], [6], and [7] provide the terminology and notation for this paper.

The following two propositions are true:

(1) Let I_1, I_2 be finite sequences of elements of the instructions of **SCM**, and let D be a finite sequence of elements of $\mathbb{Z}$, and let i_1, p_1, d_1 be natural numbers. Then every state with instruction counter on i_1, with $I_1 \frown I_2$ located from p_1, and D from d_1 is a state with instruction counter on i_1, with I_1 located from p_1, and D from d_1 and a state with instruction counter on i_1, with I_2 located from $p_1 + \operatorname{len} I_1$, and D from d_1.

(2) Let I_1, I_2 be finite sequences of elements of the instructions of **SCM**, and let i_1, p_1, d_1, k, i_2 be natural numbers, and let s be a state with instruction counter on i_1, with $I_1 \frown I_2$ located from p_1, and $\varepsilon_{\mathbb{Z}}$ from d_1, and let u be a state of **SCM**. Suppose $u = (\text{Computation}(s))(k)$ and

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$\mathbf{i}_{(i_2)} = \mathbf{IC}_u$. Then u is a state with instruction counter on i_2, with I_2 located from $p_1 + \operatorname{len} I_1$, and $\varepsilon_{\mathbb{Z}}$ from d_1.

The binary strict non empty tree construction structure $\mathrm{AE}_{\mathrm{SCM}}$ with terminals, nonterminals, and useful nonterminals is defined by the conditions (Def.1).

(Def.1) (i) The terminals of $\mathrm{AE}_{\mathrm{SCM}} = \text{Data-Loc}_{\mathrm{SCM}}$,

(ii) the nonterminals of $\mathrm{AE}_{\mathrm{SCM}} = [\!: 1, 5 :\!]$, and

(iii) for all symbols x, y, z of $\mathrm{AE}_{\mathrm{SCM}}$ holds $x \Rightarrow \langle y, z \rangle$ iff $x \in [\!: 1, 5 :\!]$.

A binary term is an element of $\mathrm{TS}(\mathrm{AE}_{\mathrm{SCM}})$.

Let n_1 be a nonterminal of $\mathrm{AE}_{\mathrm{SCM}}$ and let t_1, t_2 be binary terms. Then n_1-tree(t_1, t_2) is a binary term.

Let t be a terminal of $\mathrm{AE}_{\mathrm{SCM}}$. Then the root tree of t is a binary term.

Let t be a terminal of $\mathrm{AE}_{\mathrm{SCM}}$. The functor ${}^{@}t$ yielding a data-location is defined as follows:

(Def.2) ${}^{@}t = t$.

One can prove the following propositions:

(3) For every nonterminal n_1 of $\mathrm{AE}_{\mathrm{SCM}}$ holds $n_1 = \langle 0, 0 \rangle$ or $n_1 = \langle 0, 1 \rangle$ or $n_1 = \langle 0, 2 \rangle$ or $n_1 = \langle 0, 3 \rangle$ or $n_1 = \langle 0, 4 \rangle$.

(4) (i) $\langle 0, 0 \rangle$ is a nonterminal of $\mathrm{AE}_{\mathrm{SCM}}$,

(ii) $\langle 0, 1 \rangle$ is a nonterminal of $\mathrm{AE}_{\mathrm{SCM}}$,

(iii) $\langle 0, 2 \rangle$ is a nonterminal of $\mathrm{AE}_{\mathrm{SCM}}$,

(iv) $\langle 0, 3 \rangle$ is a nonterminal of $\mathrm{AE}_{\mathrm{SCM}}$, and

(v) $\langle 0, 4 \rangle$ is a nonterminal of $\mathrm{AE}_{\mathrm{SCM}}$.

Let t_3, t_4 be binary terms. The functor $t_3 + t_4$ yields a binary term and is defined as follows:

(Def.3) $t_3 + t_4 = \langle 0, 0 \rangle\text{-tree}(t_3, t_4)$.

The functor $t_3 - t_4$ yielding a binary term is defined as follows:

(Def.4) $t_3 - t_4 = \langle 0, 1 \rangle\text{-tree}(t_3, t_4)$.

The functor $t_3 \cdot t_4$ yields a binary term and is defined by:

(Def.5) $t_3 \cdot t_4 = \langle 0, 2 \rangle\text{-tree}(t_3, t_4)$.

The functor $t_3 \div t_4$ yields a binary term and is defined by:

(Def.6) $t_3 \div t_4 = \langle 0, 3 \rangle\text{-tree}(t_3, t_4)$.

The functor $t_3 \bmod t_4$ yielding a binary term is defined as follows:

(Def.7) $t_3 \bmod t_4 = \langle 0, 4 \rangle\text{-tree}(t_3, t_4)$.

We now state the proposition

(5) Let t_5 be a binary term. Then

(i) there exists a terminal t of $\mathrm{AE}_{\mathrm{SCM}}$ such that $t_5 =$ the root tree of t, or

(ii) there exist binary terms t_1, t_2 such that $t_5 = t_1 + t_2$ or $t_5 = t_1 - t_2$ or $t_5 = t_1 \cdot t_2$ or $t_5 = t_1 \div t_2$ or $t_5 = t_1 \bmod t_2$.

Let o be a nonterminal of $\mathrm{AE}_{\mathrm{SCM}}$ and let i, j be integers. The functor $o(i, j)$ yielding an integer is defined as follows:

(Def.8) (i) $o(i, j) = i + j$ if $o = \langle 0, 0 \rangle$,
(ii) $o(i, j) = i - j$ if $o = \langle 0, 1 \rangle$,
(iii) $o(i, j) = i \cdot j$ if $o = \langle 0, 2 \rangle$,
(iv) $o(i, j) = i \div j$ if $o = \langle 0, 3 \rangle$,
(v) $o(i, j) = i \bmod j$ if $o = \langle 0, 4 \rangle$.

Let s be a state of **SCM** and let t be a terminal of $\mathrm{AE_{SCM}}$. Then $s(t)$ is an integer.

$\mathbb{Z}$ is a non empty subset of $\mathbb{R}$.

One can verify that every element of $\mathbb{Z}$ is integer.

Let D be a non empty set, let f be a function from $\mathbb{Z}$ into D, and let x be an integer. Then $f(x)$ is an element of D.

Let s be a state of **SCM** and let t_5 be a binary term. The functor $t_5 \mathbin{@} s$ yields an integer and is defined by the condition (Def.9).

(Def.9) There exists a function f from $\mathrm{TS(AE_{SCM})}$ into $\mathbb{Z}$ such that
(i) $t_5 \mathbin{@} s = f(t_5)$,
(ii) for every terminal t of $\mathrm{AE_{SCM}}$ holds $f(\text{the root tree of } t) = s(t)$, and
(iii) for every nonterminal n_1 of $\mathrm{AE_{SCM}}$ and for all binary terms t_1, t_2 and for all symbols r_1, r_2 of $\mathrm{AE_{SCM}}$ such that $r_1 =$ the root label of t_1 and $r_2 =$ the root label of t_2 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ and for all elements x_1, x_2 of $\mathbb{Z}$ such that $x_1 = f(t_1)$ and $x_2 = f(t_2)$ holds $f(n_1\text{-tree}(t_1, t_2)) = n_1(x_1, x_2)$.

One can prove the following three propositions:

(6) For every state s of **SCM** and for every terminal t of $\mathrm{AE_{SCM}}$ holds (the root tree of t) $\mathbin{@} s = s(t)$.

(7) For every state s of **SCM** and for every nonterminal n_1 of $\mathrm{AE_{SCM}}$ and for all binary terms t_1, t_2 holds $(n_1\text{-tree}(t_1, t_2)) \mathbin{@} s = n_1(t_1 \mathbin{@} s, t_2 \mathbin{@} s)$.

(8) Let s be a state of **SCM** and let t_1, t_2 be binary terms. Then $(t_1 + t_2) \mathbin{@} s = (t_1 \mathbin{@} s) + (t_2 \mathbin{@} s)$ and $(t_1 - t_2) \mathbin{@} s = (t_1 \mathbin{@} s) - (t_2 \mathbin{@} s)$ and $t_1 \cdot t_2 \mathbin{@} s = (t_1 \mathbin{@} s) \cdot (t_2 \mathbin{@} s)$ and $(t_1 \div t_2) \mathbin{@} s = (t_1 \mathbin{@} s) \div (t_2 \mathbin{@} s)$ and $(t_1 \bmod t_2) \mathbin{@} s = (t_1 \mathbin{@} s) \bmod (t_2 \mathbin{@} s)$.

Let n_1 be a nonterminal of $\mathrm{AE_{SCM}}$ and let n be a natural number. The functor $\mathrm{Selfwork}(n_1, n)$ yielding an element of (the instructions of **SCM qua** set)* is defined as follows:

(Def.10) (i) $\mathrm{Selfwork}(n_1, n) = \langle \mathrm{AddTo}(\mathbf{d}_n, \mathbf{d}_{n+1}) \rangle$ if $n_1 = \langle 0, 0 \rangle$,
(ii) $\mathrm{Selfwork}(n_1, n) = \langle \mathrm{SubFrom}(\mathbf{d}_n, \mathbf{d}_{n+1}) \rangle$ if $n_1 = \langle 0, 1 \rangle$,
(iii) $\mathrm{Selfwork}(n_1, n) = \langle \mathrm{MultBy}(\mathbf{d}_n, \mathbf{d}_{n+1}) \rangle$ if $n_1 = \langle 0, 2 \rangle$,
(iv) $\mathrm{Selfwork}(n_1, n) = \langle \mathrm{Divide}(\mathbf{d}_n, \mathbf{d}_{n+1}) \rangle$ if $n_1 = \langle 0, 3 \rangle$,
(v) $\mathrm{Selfwork}(n_1, n) = \langle \mathrm{Divide}(\mathbf{d}_n, \mathbf{d}_{n+1}), \mathbf{d}_n{:=}\mathbf{d}_{n+1} \rangle$ if $n_1 = \langle 0, 4 \rangle$.

Let t_5 be a binary term and let a_1 be a natural number. The functor $\mathrm{Compile}(t_5, a_1)$ yielding a finite sequence of elements of the instructions of **SCM** is defined by the condition (Def.11).

(Def.11) There exists a function f from $\mathrm{TS(AE_{SCM})}$ into ((the instructions of **SCM qua** set)*)$^{\mathbb{N}}$ such that
(i) $\mathrm{Compile}(t_5, a_1) = (f(t_5)$ **qua** element of ((the instructions of **SCM**

qua set)*)$^{\mathbb{N}}$)(a_1),

(ii) for every terminal t of $\mathrm{AE}_{\mathrm{SCM}}$ there exists a function g from $\mathbb{N}$ into (the instructions of **SCM qua** set)* such that $g = f$(the root tree of t) and for every natural number n holds $g(n) = \langle \mathbf{d}_n{:=}^{@}t \rangle$, and

(iii) for every nonterminal n_1 of $\mathrm{AE}_{\mathrm{SCM}}$ and for all binary terms t_3, t_4 and for all symbols r_1, r_2 of $\mathrm{AE}_{\mathrm{SCM}}$ such that $r_1 =$ the root label of t_3 and $r_2 =$ the root label of t_4 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ there exist functions g, f_1, f_2 from $\mathbb{N}$ into (the instructions of **SCM qua** set)* such that $g = f(n_1\text{-tree}(t_3, t_4))$ and $f_1 = f(t_3)$ and $f_2 = f(t_4)$ and for every natural number n holds $g(n) = f_1(n) \frown f_2(n+1) \frown \mathrm{Selfwork}(n_1, n)$.

One can prove the following propositions:

(9) For every terminal t of $\mathrm{AE}_{\mathrm{SCM}}$ and for every natural number n holds Compile(the root tree of t, n) $= \langle \mathbf{d}_n{:=}^{@}t \rangle$.

(10) Let n_1 be a nonterminal of $\mathrm{AE}_{\mathrm{SCM}}$, and let t_3, t_4 be binary terms, and let n be a natural number, and let r_1, r_2 be symbols of $\mathrm{AE}_{\mathrm{SCM}}$. Suppose $r_1 =$ the root label of t_3 and $r_2 =$ the root label of t_4 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$. Then $\mathrm{Compile}(n_1\text{-tree}(t_3, t_4), n) = (\mathrm{Compile}(t_3, n)) \frown \mathrm{Compile}(t_4, n+1) \frown \mathrm{Selfwork}(n_1, n)$.

Let t be a terminal of $\mathrm{AE}_{\mathrm{SCM}}$. The functor $\mathbf{d}^{-1}(t)$ yielding a natural number is defined as follows:

(Def.12) $\mathbf{d}_{\mathbf{d}^{-1}(t)} = t$.

Let n_2, n_3 be natural numbers. Then $\max(n_2, n_3)$ is a natural number.

Let t_5 be a binary term. The functor $\max_{\mathrm{DL}}(t_5)$ yielding a natural number is defined by the condition (Def.13).

(Def.13) There exists a function f from $\mathrm{TS}(\mathrm{AE}_{\mathrm{SCM}})$ into $\mathbb{N}$ such that

(i) $\max_{\mathrm{DL}}(t_5) = f(t_5)$,

(ii) for every terminal t of $\mathrm{AE}_{\mathrm{SCM}}$ holds f(the root tree of t) $= \mathbf{d}^{-1}(t)$, and

(iii) for every nonterminal n_1 of $\mathrm{AE}_{\mathrm{SCM}}$ and for all binary terms t_1, t_2 and for all symbols r_1, r_2 of $\mathrm{AE}_{\mathrm{SCM}}$ such that $r_1 =$ the root label of t_1 and $r_2 =$ the root label of t_2 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ and for all natural numbers x_1, x_2 such that $x_1 = f(t_1)$ and $x_2 = f(t_2)$ holds $f(n_1\text{-tree}(t_1, t_2)) = \max(x_1, x_2)$.

One can prove the following propositions:

(11) For every terminal t of $\mathrm{AE}_{\mathrm{SCM}}$ holds $\max_{\mathrm{DL}}$(the root tree of t) $= \mathbf{d}^{-1}(t)$.

(12) For every nonterminal n_1 of $\mathrm{AE}_{\mathrm{SCM}}$ and for all binary terms t_1, t_2 holds $\max_{\mathrm{DL}}(n_1\text{-tree}(t_1, t_2)) = \max(\max_{\mathrm{DL}}(t_1), \max_{\mathrm{DL}}(t_2))$.

(13) Let t_5 be a binary term and let s_1, s_2 be states of **SCM**. Suppose that for every natural number d_2 such that $d_2 \leq \max_{\mathrm{DL}}(t_5)$ holds $s_1(\mathbf{d}_{(d_2)}) = s_2(\mathbf{d}_{(d_2)})$. Then $t_5 \ {}^{@}\ s_1 = t_5 \ {}^{@}\ s_2$.

(14) Let t_5 be a binary term, and let a_1, n, k be natural numbers, and let s be a state with instruction counter on n, with $\mathrm{Compile}(t_5, a_1)$ located

from n, and $\varepsilon_{\mathbb{Z}}$ from k. Suppose $a_1 > \max_{\rm DL}(t_5)$. Then there exists a natural number i and there exists a state u of **SCM** such that

(i) $u = ({\rm Computation}(s))(i+1)$,
(ii) $i+1 = {\rm len\,Compile}(t_5, a_1)$,
(iii) $\mathbf{IC}_{({\rm Computation}(s))(i)} = \mathbf{i}_{n+i}$,
(iv) $\mathbf{IC}_u = \mathbf{i}_{n+(i+1)}$,
(v) $u(\mathbf{d}_{(a_1)}) = t_5\ {}^{@}\ s$, and
(vi) for every natural number d_2 such that $d_2 < a_1$ holds $s(\mathbf{d}_{(d_2)}) = u(\mathbf{d}_{(d_2)})$.

(15) Let t_5 be a binary term, and let a_1, n, k be natural numbers, and let s be a state with instruction counter on n, with $({\rm Compile}(t_5, a_1)) \frown \langle \mathbf{halt_{SCM}} \rangle$ located from n, and $\varepsilon_{\mathbb{Z}}$ from k. Suppose $a_1 > \max_{\rm DL}(t_5)$. Then s is halting and $({\rm Result}(s))(\mathbf{d}_{(a_1)}) = t_5\ {}^{@}\ s$ and the complexity of $s = {\rm len\,Compile}(t_5, a_1)$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek. Joining of decorated trees. *Formalized Mathematics*, 4(**1**):77–82, 1993.
[3] Grzegorz Bancerek. König's lemma. *Formalized Mathematics*, 2(**3**):397–402, 1991.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Grzegorz Bancerek and Piotr Rudnicki. Development of terminology for **SCM**. *Formalized Mathematics*, 4(**1**):61–67, 1993.
[6] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on binary trees. *Formalized Mathematics*, 5(**1**):9–13, 1996.
[7] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Formalized Mathematics*, 4(**1**):91–101, 1993.
[8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[11] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar - part 1. *Formalized Mathematics*, 2(**5**):683–687, 1991.
[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[13] Yatsuka Nakamura and Andrzej Trybulec. A mathematical model of CPU. *Formalized Mathematics*, 3(**2**):151–160, 1992.
[14] Yatsuka Nakamura and Andrzej Trybulec. On a mathematical model of programs. *Formalized Mathematics*, 3(**2**):241–250, 1992.
[15] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[18] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[19] Andrzej Trybulec and Yatsuka Nakamura. Some remarks on the simple concrete model of computer. *Formalized Mathematics*, 4(**1**):51–56, 1993.

[20] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[21] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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Some Properties of the Intervals

Józef Białas
Łódź University

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The papers [8], [10], [4], [5], [6], [1], [2], [3], [7], and [9] provide the terminology and notation for this paper.

The scheme *FunctXD YD* concerns a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a function F from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $\mathcal{P}[x, F(x)]$

provided the following condition is satisfied:

- For every element x of $\mathcal{A}$ there exists an element y of $\mathcal{B}$ such that $\mathcal{P}[x, y]$.

Let X, Y be non empty sets. Note that Y^X is non empty.

We now state a number of propositions:

(1) There exists a function F from $\mathbb{N}$ into $[\!:\mathbb{N}, \mathbb{N}:\!]$ such that F is one-to-one and $\operatorname{dom} F = \mathbb{N}$ and $\operatorname{rng} F = [\!:\mathbb{N}, \mathbb{N}:\!]$.

(2) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative holds $0_{\overline{\mathbb{R}}} \leq \sum F$.

(3) Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and let x be a *Real number*. Suppose there exists a natural number n such that $x \leq F(n)$ and F is non-negative. Then $x \leq \sum F$.

(4) For every *Real number* x such that there exists a *Real number* y such that $y < x$ holds $x \neq -\infty$.

(5) For every *Real number* x such that there exists a *Real number* y such that $x < y$ holds $x \neq +\infty$.

(6) For all *Real numbers* x, y holds $x \leq y$ iff $x < y$ or $x = y$.

(7) Let x, y be *Real numbers* and let p, q be real numbers. If $x = p$ and $y = q$, then $p \leq q$ iff $x \leq y$.

(8) For all *Real numbers* x, y such that x is a real number holds $(y-x)+x = y$ and $(y+x)-x = y$.

(9) For all *Real numbers* x, y such that $x \in \mathbb{R}$ holds $x + y = y + x$.

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(10) For all *Real numbers* x, y, z such that $z \in \mathbb{R}$ and $y < x$ holds $(z + x) - (z + y) = x - y$.

(11) For all *Real numbers* x, y, z such that $z \in \mathbb{R}$ and $x \leq y$ holds $z+x \leq z+y$ and $x + z \leq y + z$ and $x - z \leq y - z$.

(12) For all *Real numbers* x, y, z such that $z \in \mathbb{R}$ and $x < y$ holds $z+x < z+y$ and $x + z < y + z$ and $x - z < y - z$.

Let x be a real number. The functor $\overline{\mathbb{R}}(x)$ yields a *Real number* and is defined as follows:

(Def.1) $\overline{\mathbb{R}}(x) = x$.

The following propositions are true:

(13) For all real numbers x, y holds $x \leq y$ iff $\overline{\mathbb{R}}(x) \leq \overline{\mathbb{R}}(y)$.

(14) For all real numbers x, y holds $x < y$ iff $\overline{\mathbb{R}}(x) < \overline{\mathbb{R}}(y)$.

(15) For all *Real numbers* x, y, z such that $x < y$ and $y < z$ holds y is a real number.

(16) Let x, y, z be *Real numbers*. Suppose x is a real number and z is a real number and $x \leq y$ and $y \leq z$. Then y is a real number.

(17) For all *Real numbers* x, y, z such that x is a real number and $x \leq y$ and $y < z$ holds y is a real number.

(18) For all *Real numbers* x, y, z such that $x < y$ and $y \leq z$ and z is a real number holds y is a real number.

(19) For all *Real numbers* x, y such that $0_{\overline{\mathbb{R}}} < x$ and $x < y$ holds $0_{\overline{\mathbb{R}}} < y - x$.

(20) For all *Real numbers* x, y, z such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq z$ and $z + x < y$ holds $z < y - x$.

(21) For every *Real number* x holds $x - 0_{\overline{\mathbb{R}}} = x$.

(22) For all *Real numbers* x, y, z such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq z$ and $z + x < y$ holds $z \leq y$.

(23) For every *Real number* x such that $0_{\overline{\mathbb{R}}} < x$ there exists a *Real number* y such that $0_{\overline{\mathbb{R}}} < y$ and $y < x$.

(24) Let x, z be *Real numbers*. Suppose $0_{\overline{\mathbb{R}}} < x$ and $x < z$. Then there exists a *Real number* y such that $0_{\overline{\mathbb{R}}} < y$ and $x + y < z$ and $y \in \mathbb{R}$.

(25) Let x, z be *Real numbers*. Suppose $0_{\overline{\mathbb{R}}} \leq x$ and $x < z$. Then there exists a *Real number* y such that $0_{\overline{\mathbb{R}}} < y$ and $x + y < z$ and $y \in \mathbb{R}$.

(26) For every *Real number* x such that $0_{\overline{\mathbb{R}}} < x$ there exists a *Real number* y such that $0_{\overline{\mathbb{R}}} < y$ and $y + y < x$.

Let x be a *Real number*. Let us assume that $0_{\overline{\mathbb{R}}} < x$. The functor $\mathrm{Seg}\, x$ yields a non empty subset of $\overline{\mathbb{R}}$ and is defined by:

(Def.2) For every *Real number* y holds $y \in \mathrm{Seg}\, x$ iff $0_{\overline{\mathbb{R}}} < y$ and $y + y < x$.

Let x be a *Real number*. Let us assume that $0_{\overline{\mathbb{R}}} < x$. The functor $\mathrm{len}\, x$ yielding a *Real number* is defined as follows:

(Def.3) $\mathrm{len}\, x = \sup \mathrm{Seg}\, x$.

Next we state several propositions:

(27) For every *Real number* x such that $0_{\overline{\mathbb{R}}} < x$ holds $0_{\overline{\mathbb{R}}} < \operatorname{len} x$.

(28) For every *Real number* x such that $0_{\overline{\mathbb{R}}} < x$ holds $\operatorname{len} x \leq x$.

(29) For every *Real number* x such that $0_{\overline{\mathbb{R}}} < x$ and $x < +\infty$ holds $\operatorname{len} x$ is a real number.

(30) For every *Real number* x such that $0_{\overline{\mathbb{R}}} < x$ holds $\operatorname{len} x + \operatorname{len} x \leq x$.

(31) Let e_1 be a *Real number*. Suppose $0_{\overline{\mathbb{R}}} < e_1$. Then there exists a function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every natural number n holds $0_{\overline{\mathbb{R}}} < F(n)$ and $\sum F < e_1$.

(32) Let e_1 be a *Real number* and let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose $0_{\overline{\mathbb{R}}} < e_1$ and $\inf X$ is a real number. Then there exists a *Real number* x such that $x \in X$ and $x < \inf X + e_1$.

(33) Let e_1 be a *Real number* and let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose $0_{\overline{\mathbb{R}}} < e_1$ and $\sup X$ is a real number. Then there exists a *Real number* x such that $x \in X$ and $\sup X - e_1 < x$.

(34) Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose F is non-negative and $\sum F < +\infty$. Let n be a natural number. Then $F(n) \in \mathbb{R}$.

$-\infty$ is a *Real number*.

$+\infty$ is a *Real number*.

We now state a number of propositions:

(35) $\mathbb{R}$ is an interval and $\mathbb{R} =]-\infty, +\infty[$ and $\mathbb{R} = [-\infty, +\infty]$ and $\mathbb{R} = [-\infty, +\infty[$ and $\mathbb{R} =]-\infty, +\infty]$.

(36) For all *Real numbers* a, b such that $b = -\infty$ holds $]a, b[= \emptyset$ and $[a, b] = \emptyset$ and $[a, b[= \emptyset$ and $]a, b] = \emptyset$.

(37) For all *Real numbers* a, b such that $a = +\infty$ holds $]a, b[= \emptyset$ and $[a, b] = \emptyset$ and $[a, b[= \emptyset$ and $]a, b] = \emptyset$.

(38) Let A be an interval and let a, b be *Real numbers*. Suppose $A =]a, b[$. Let c, d be real numbers. Suppose $c \in A$ and $d \in A$. Let e be a real number. If $c \leq e$ and $e \leq d$, then $e \in A$.

(39) Let A be an interval and let a, b be *Real numbers*. Suppose $A = [a, b]$. Let c, d be real numbers. Suppose $c \in A$ and $d \in A$. Let e be a real number. If $c \leq e$ and $e \leq d$, then $e \in A$.

(40) Let A be an interval and let a, b be *Real numbers*. Suppose $A =]a, b]$. Let c, d be real numbers. Suppose $c \in A$ and $d \in A$. Let e be a real number. If $c \leq e$ and $e \leq d$, then $e \in A$.

(41) Let A be an interval and let a, b be *Real numbers*. Suppose $A = [a, b[$. Let c, d be real numbers. Suppose $c \in A$ and $d \in A$. Let e be a real number. If $c \leq e$ and $e \leq d$, then $e \in A$.

(42) Let A be a non empty subset of $\overline{\mathbb{R}}$ and let m, M be *Real numbers*. Suppose $m = \inf A$ and $M = \sup A$. Suppose that

(i) for all real numbers c, d such that $c \in A$ and $d \in A$ and for every real number e such that $c \leq e$ and $e \leq d$ holds $e \in A$,

(ii) $m \notin A$, and

(iii) $M \notin A$.
Then $A =]m, M[$.

(43) Let A be a non empty subset of $\overline{\mathbb{R}}$ and let m, M be *Real numbers*. Suppose $m = \inf A$ and $M = \sup A$. Suppose that
(i) for all real numbers c, d such that $c \in A$ and $d \in A$ and for every real number e such that $c \leq e$ and $e \leq d$ holds $e \in A$,
(ii) $m \in A$,
(iii) $M \in A$, and
(iv) $A \subseteq \mathbb{R}$.
Then $A = [m, M]$.

(44) Let A be a non empty subset of $\overline{\mathbb{R}}$ and let m, M be *Real numbers*. Suppose $m = \inf A$ and $M = \sup A$. Suppose that
(i) for all real numbers c, d such that $c \in A$ and $d \in A$ and for every real number e such that $c \leq e$ and $e \leq d$ holds $e \in A$,
(ii) $m \in A$,
(iii) $M \notin A$, and
(iv) $A \subseteq \mathbb{R}$.
Then $A = [m, M[$.

(45) Let A be a non empty subset of $\overline{\mathbb{R}}$ and let m, M be *Real numbers*. Suppose $m = \inf A$ and $M = \sup A$. Suppose that
(i) for all real numbers c, d such that $c \in A$ and $d \in A$ and for every real number e such that $c \leq e$ and $e \leq d$ holds $e \in A$,
(ii) $m \notin A$,
(iii) $M \in A$, and
(iv) $A \subseteq \mathbb{R}$.
Then $A =]m, M]$.

(46) Let A be a subset of $\mathbb{R}$. Then A is an interval if and only if for all real numbers a, b such that $a \in A$ and $b \in A$ and for every real number c such that $a \leq c$ and $c \leq b$ holds $c \in A$.

Let A, B be intervals. Then $A \cup B$ is a subset of $\mathbb{R}$.

Next we state the proposition

(47) For all intervals A, B such that $A \cap B \neq \emptyset$ holds $A \cup B$ is an interval.

Let A be an interval. Let us assume that $A \neq \emptyset$. The functor $\inf A$ yields a *Real number* and is defined as follows:

(Def.4) There exists a *Real number* b such that $\inf A \leq b$ but $A =]\inf A, b[$ or $A =]\inf A, b]$ or $A = [\inf A, b]$ or $A = [\inf A, b[$.

Let A be an interval. Let us assume that $A \neq \emptyset$. The functor $\sup A$ yielding a *Real number* is defined as follows:

(Def.5) There exists a *Real number* a such that $a \leq \sup A$ but $A =]a, \sup A[$ or $A =]a, \sup A]$ or $A = [a, \sup A]$ or $A = [a, \sup A[$.

Next we state a number of propositions:

(48) For every interval A such that A is open interval and $A \neq \emptyset$ holds $\inf A \leq \sup A$ and $A =]\inf A, \sup A[$.

(49) For every interval A such that A is closed interval and $A \neq \emptyset$ holds $\inf A \leq \sup A$ and $A = [\inf A, \sup A]$.

(50) For every interval A such that A is right open interval and $A \neq \emptyset$ holds $\inf A \leq \sup A$ and $A = [\inf A, \sup A[$.

(51) For every interval A such that A is left open interval and $A \neq \emptyset$ holds $\inf A \leq \sup A$ and $A =]\inf A, \sup A]$.

(52) For every interval A such that $A \neq \emptyset$ holds $\inf A \leq \sup A$ but $A =]\inf A, \sup A[$ or $A =]\inf A, \sup A]$ or $A = [\inf A, \sup A]$ or $A = [\inf A, \sup A[$.

(53) For all intervals A, B such that $A = \emptyset$ or $B = \emptyset$ holds $A \cup B$ is an interval.

(54) For every interval A and for every real number a such that $a \in A$ holds $\inf A \leq \overline{\mathbb{R}}(a)$ and $\overline{\mathbb{R}}(a) \leq \sup A$.

(55) For all intervals A, B and for all real numbers a, b such that $a \in A$ and $b \in B$ holds if $\sup A \leq \inf B$, then $a \leq b$.

(56) For every interval A and for every *Real number* a such that $a \in A$ holds $\inf A \leq a$ and $a \leq \sup A$.

(57) For every interval A such that $A \neq \emptyset$ and for every *Real number* a such that $\inf A < a$ and $a < \sup A$ holds $a \in A$.

(58) For all intervals A, B such that $\sup A = \inf B$ but $\sup A \in A$ or $\inf B \in B$ holds $A \cup B$ is an interval.

Let A be a subset of $\mathbb{R}$ and let x be a real number. The functor $x + A$ yields a subset of $\mathbb{R}$ and is defined by:

(Def.6) For every real number y holds $y \in x + A$ iff there exists a real number z such that $z \in A$ and $y = x + z$.

One can prove the following propositions:

(59) For every subset A of $\mathbb{R}$ and for every real number x holds $-x+(x+A) = A$.

(60) For every real number x and for every subset A of $\mathbb{R}$ such that $A = \mathbb{R}$ holds $x + A = A$.

(61) For every real number x holds $x + \emptyset = \emptyset$.

(62) For every interval A and for every real number x holds A is open interval iff $x + A$ is open interval.

(63) For every interval A and for every real number x holds A is closed interval iff $x + A$ is closed interval.

(64) Let A be an interval and let x be a real number. Then A is right open interval if and only if $x + A$ is right open interval.

(65) Let A be an interval and let x be a real number. Then A is left open interval if and only if $x + A$ is left open interval.

(66) For every interval A and for every real number x holds $x + A$ is an interval.

Let A be an interval and let x be a real number. Note that $x + A$ is interval. The following proposition is true

(67) For every interval A and for every real number x holds $\mathrm{vol}(A) = \mathrm{vol}(x + A)$.

References

[1] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):163–171, 1991.
[2] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.
[3] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[7] Białas Józef. Properties of the intervals of real numbers. *Formalized Mathematics*, 3(**2**):263–269, 1992.
[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[9] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[10] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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Binary Arithmetics, Addition and Subtraction of Integers

Yasuho Mizuhara
Shinshu University
Information Engineering Dept.
Nagano

Takaya Nishiyama
Shinshu University
Information Engineering Dept.
Nagano

Summary. This article is a continuation of [6] and presents the concepts of binary arithmetic operations for integers. There is introduced 2's complement representation of integers and natural numbers to integers are expanded. The binary addition and subtraction for integers are defined and theorems on the relationship between binary and numerical operations presented.

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The notation and terminology used here are introduced in the following papers: [8], [5], [4], [9], [11], [7], [2], [1], [3], [10], and [6].

Let X be a non empty set, let D be a non empty subset of X, let x, y be arbitrary, and let a, b be elements of D. Then $(x = y \to a, b)$ is an element of D.

We follow the rules: i will be a natural number, n will be a non empty natural number, and x, y, z_1, z_2 will be tuples of n and *Boolean*.

Let us consider n. The functor $\mathrm{Bin1}(n)$ yielding a tuple of n and *Boolean* is defined by:

(Def.1) For every i such that $i \in \mathrm{Seg}\, n$ holds $\pi_i \mathrm{Bin1}(n) = (i = 1 \to \mathit{true}, \mathit{false})$.

Let us consider n, x. The functor $\mathrm{Neg2}(x)$ yielding a tuple of n and *Boolean* is defined by:

(Def.2) $\mathrm{Neg2}(x) = \neg x + \mathrm{Bin1}(n)$.

Let us consider n, x. The functor $\mathrm{Intval}(x)$ yielding an integer is defined by:

(Def.3) (i) $\mathrm{Intval}(x) = \mathrm{Absval}(x)$ if $\pi_n x = \mathit{false}$,

(ii) $\mathrm{Intval}(x) = \mathrm{Absval}(x) - (\text{the } n\text{-th power of } 2)$, otherwise.

Let us consider n, z_1, z_2. The functor $\mathrm{Int\_add\_ovfl}(z_1, z_2)$ yields an element of *Boolean* and is defined by:

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(Def.4) $\mathrm{Int\_add\_ovfl}(z_1, z_2) = \neg\pi_n z_1 \wedge \neg\pi_n z_2 \wedge \pi_n \mathrm{carry}(z_1, z_2)$.

Let us consider n, z_1, z_2. The functor $\mathrm{Int\_add\_udfl}(z_1, z_2)$ yields an element of *Boolean* and is defined by:

(Def.5) $\mathrm{Int\_add\_udfl}(z_1, z_2) = \pi_n z_1 \wedge \pi_n z_2 \wedge \neg\pi_n \mathrm{carry}(z_1, z_2)$.

The following propositions are true:

(1) For every tuple z_1 of 1 and *Boolean* such that $z_1 = \langle \mathit{false} \rangle$ holds $\mathrm{Absval}(z_1) = 0$.

(2) For every tuple z_1 of 1 and *Boolean* such that $z_1 = \langle \mathit{true} \rangle$ holds $\mathrm{Absval}(z_1) = 1$.

(3) For every tuple z_1 of 2 and *Boolean* such that $z_1 = \langle \mathit{false} \rangle \frown \langle \mathit{false} \rangle$ holds $\mathrm{Intval}(z_1) = 0$.

(4) For every tuple z_1 of 2 and *Boolean* such that $z_1 = \langle \mathit{true} \rangle \frown \langle \mathit{false} \rangle$ holds $\mathrm{Intval}(z_1) = 1$.

(5) For every tuple z_1 of 2 and *Boolean* such that $z_1 = \langle \mathit{false} \rangle \frown \langle \mathit{true} \rangle$ holds $\mathrm{Intval}(z_1) = -2$.

(6) For every tuple z_1 of 2 and *Boolean* such that $z_1 = \langle \mathit{true} \rangle \frown \langle \mathit{true} \rangle$ holds $\mathrm{Intval}(z_1) = -1$.

(7) For every i such that $i \in \mathrm{Seg}\, n$ and $i = 1$ holds $\pi_i \mathrm{Bin1}(n) = \mathit{true}$.

(8) For every i such that $i \in \mathrm{Seg}\, n$ and $i \neq 1$ holds $\pi_i \mathrm{Bin1}(n) = \mathit{false}$.

(9) For every n holds $\mathrm{Bin1}(n+1) = (\mathrm{Bin1}(n)) \frown \langle \mathit{false} \rangle$.

(10) For every n holds $\mathrm{Intval}((\mathrm{Bin1}(n)) \frown \langle \mathit{false} \rangle) = 1$.

(11) For every n and for every tuple z of n and *Boolean* and for every element d of *Boolean* holds $\neg(z \frown \langle d \rangle) = (\neg z) \frown \langle \neg d \rangle$.

(12) Given n, and let z be a tuple of n and *Boolean*, and let d be an element of *Boolean*. Then $\mathrm{Intval}(z \frown \langle d \rangle) = \mathrm{Absval}(z) - ((d = \mathit{false} \rightarrow 0, \text{the } n\text{-th power of } 2)$ **qua** natural number).

(13) Given n, and let z_1, z_2 be tuples of n and *Boolean*, and let d_1, d_2 be elements of *Boolean*. Then $(\mathrm{Intval}(z_1 \frown \langle d_1 \rangle + z_2 \frown \langle d_2 \rangle) + (\mathrm{Int\_add\_ovfl}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = \mathit{false} \rightarrow 0, \text{the } n+1\text{-th power of } 2)) - (\mathrm{Int\_add\_udfl}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = \mathit{false} \rightarrow 0, \text{the } n+1\text{-th power of } 2) = \mathrm{Intval}(z_1 \frown \langle d_1 \rangle) + \mathrm{Intval}(z_2 \frown \langle d_2 \rangle)$.

(14) Given n, and let z_1, z_2 be tuples of n and *Boolean*, and let d_1, d_2 be elements of *Boolean*. Then $\mathrm{Intval}(z_1 \frown \langle d_1 \rangle + z_2 \frown \langle d_2 \rangle) = ((\mathrm{Intval}(z_1 \frown \langle d_1 \rangle) + \mathrm{Intval}(z_2 \frown \langle d_2 \rangle)) - (\mathrm{Int\_add\_ovfl}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = \mathit{false} \rightarrow 0, \text{the } n+1\text{-th power of } 2)) + (\mathrm{Int\_add\_udfl}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = \mathit{false} \rightarrow 0, \text{the } n+1\text{-th power of } 2)$.

(15) For every n and for every tuple x of n and *Boolean* holds $\mathrm{Absval}(\neg x) = (-\mathrm{Absval}(x) + (\text{the } n\text{-th power of } 2)) - 1$.

(16) For every n and for every tuple z of n and *Boolean* and for every element d of *Boolean* holds $\mathrm{Neg2}(z \frown \langle d \rangle) = (\mathrm{Neg2}(z)) \frown \langle \neg d \oplus \mathrm{add\_ovfl}(\neg z, \mathrm{Bin1}(n)) \rangle$.

(17) Given n, and let z be a tuple of n and *Boolean*, and let d be an element of *Boolean*. Then $\mathrm{Intval}(\mathrm{Neg2}(z \frown \langle d \rangle)) + (\mathrm{Int\_add\_ovfl}(\neg(z \frown \langle d \rangle), \mathrm{Bin1}(n+1)) = \mathit{false} \to 0, \text{the } n+1\text{-th power of } 2) = -\mathrm{Intval}(z \frown \langle d \rangle)$.

(18) For every n and for every tuple z of n and *Boolean* and for every element d of *Boolean* holds $\mathrm{Neg2}(\mathrm{Neg2}(z \frown \langle d \rangle)) = z \frown \langle d \rangle$.

Let us consider n, x, y. The functor $x - y$ yielding a tuple of n and *Boolean* is defined as follows:

(Def.6) For every i such that $i \in \mathrm{Seg}\, n$ holds $\pi_i(x - y) = \pi_i x \oplus \pi_i \mathrm{Neg2}(y) \oplus \pi_i \mathrm{carry}(x, \mathrm{Neg2}(y))$.

One can prove the following three propositions:

(19) For every n and for all tuples x, y of n and *Boolean* holds $x - y = x + \mathrm{Neg2}(y)$.

(20) For every n and for all tuples z_1, z_2 of n and *Boolean* and for all elements d_1, d_2 of *Boolean* holds $z_1 \frown \langle d_1 \rangle - z_2 \frown \langle d_2 \rangle = (z_1 + \mathrm{Neg2}(z_2)) \frown \langle d_1 \oplus \neg d_2 \oplus \mathrm{add\_ovfl}(\neg z_2, \mathrm{Bin1}(n)) \oplus \mathrm{add\_ovfl}(z_1, \mathrm{Neg2}(z_2)) \rangle$.

(21) Given n, and let z_1, z_2 be tuples of n and *Boolean*, and let d_1, d_2 be elements of *Boolean*. Then $((\mathrm{Intval}(z_1 \frown \langle d_1 \rangle - z_2 \frown \langle d_2 \rangle) + (\mathrm{Int\_add\_ovfl}(z_1 \frown \langle d_1 \rangle, \mathrm{Neg2}(z_2 \frown \langle d_2 \rangle)) = \mathit{false} \to 0, \text{the } n+1\text{-th power of } 2)) - (\mathrm{Int\_add\_udfl}(z_1 \frown \langle d_1 \rangle, \mathrm{Neg2}(z_2 \frown \langle d_2 \rangle)) = \mathit{false} \to 0, \text{the } n+1\text{-th power of } 2)) + (\mathrm{Int\_add\_ovfl}(\neg(z_2 \frown \langle d_2 \rangle), \mathrm{Bin1}(n+1)) = \mathit{false} \to 0, \text{the } n+1\text{-th power of } 2) = \mathrm{Intval}(z_1 \frown \langle d_1 \rangle) - \mathrm{Intval}(z_2 \frown \langle d_2 \rangle)$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[6] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.

[7] Konrad Raczkowski and Andrzej Nędzusiak. Serieses. *Formalized Mathematics*, 2(**4**):449–452, 1991.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[9] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

[10] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[11] Edmund Woronowicz. Many–argument relations. *Formalized Mathematics*, 1(**4**):733–737, 1990.

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Boolean Properties of Lattices

Agnieszka Julia Marasik
Warsaw University
Białystok

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The article [1] provides the terminology and notation for this paper.

1. General lattice

We follow the rules: L will be a lattice and X, Y, Z, V will be elements of the carrier of L.

Let us consider L, X, Y. The functor $X \setminus Y$ yielding an element of the carrier of L is defined by:

(Def.1) $X \setminus Y = X \sqcap Y^{\rm c}$.

Let us consider L, X, Y. The functor $X \dot{-} Y$ yields an element of the carrier of L and is defined by:

(Def.2) $X \dot{-} Y = (X \setminus Y) \sqcup (Y \setminus X)$.

Let us consider L, X, Y. Let us observe that $X = Y$ if and only if:

(Def.3) $X \sqsubseteq Y$ and $Y \sqsubseteq X$.

Let us consider L, X, Y. We say that X meets Y if and only if:

(Def.4) $X \sqcap Y \neq \bot_L$.

We introduce X misses Y as an antonym of X meets Y.

We now state a number of propositions:

(1) $X \sqsubseteq X \sqcup Y$ and $Y \sqsubseteq X \sqcup Y$.

(3)[1] If $X \sqcup Y \sqsubseteq Z$, then $X \sqsubseteq Z$ and $Y \sqsubseteq Z$.

(4) $X \sqcap Y \sqsubseteq X \sqcup Z$.

(5) If $X \sqsubseteq Y$, then $X \sqcap Z \sqsubseteq Y \sqcap Z$ and $Z \sqcap X \sqsubseteq Z \sqcap Y$.

(6) If $X \sqsubseteq Z$, then $X \setminus Y \sqsubseteq Z$.

[1]The proposition (2) has been removed.

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(7) If $X \sqsubseteq Y$, then $X \setminus Z \sqsubseteq Y \setminus Z$.

(8) $X \setminus Y \sqsubseteq X$.

(9) $X \setminus Y \sqsubseteq X \dot{-} Y$.

(10) If $X \setminus Y \sqsubseteq Z$ and $Y \setminus X \sqsubseteq Z$, then $X \dot{-} Y \sqsubseteq Z$.

(11) $X = Y \sqcup Z$ iff $Y \sqsubseteq X$ and $Z \sqsubseteq X$ and for every V such that $Y \sqsubseteq V$ and $Z \sqsubseteq V$ holds $X \sqsubseteq V$.

(12) $X = Y \sqcap Z$ iff $X \sqsubseteq Y$ and $X \sqsubseteq Z$ and for every V such that $V \sqsubseteq Y$ and $V \sqsubseteq Z$ holds $V \sqsubseteq X$.

(13) If $X \sqcup Y = Y$ or $Y \sqcup X = Y$, then $X \sqsubseteq Y$.

(14) $X \sqcap (Y \setminus Z) = X \sqcap Y \setminus Z$.

(15) If X meets Y, then Y meets X.

(16) X meets X iff $X \neq \bot_L$.

(17) $X \dot{-} Y = Y \dot{-} X$.

2. Modular lattice

In the sequel L will denote a modular lattice and X, Y will denote elements of the carrier of L.

The following three propositions are true:

(18) If $Y \sqsubseteq X$ and $X \sqcap Y = \bot_L$, then $Y = \bot_L$.

(20)[2] If $X \sqsubseteq Y$, then $X \sqcup Y = Y$ and $Y \sqcup X = Y$.

(21) If X misses Y, then Y misses X.

3. Distributive lattice

In the sequel L will denote a distributive lattice and X, Y, Z will denote elements of the carrier of L.

Next we state three propositions:

(22) If $X \sqcap Y \sqcup X \sqcap Z = X$, then $X \sqsubseteq Y \sqcup Z$.

(23) $X \sqcap Y \sqcup Y \sqcap Z \sqcup Z \sqcap X = (X \sqcup Y) \sqcap (Y \sqcup Z) \sqcap (Z \sqcup X)$.

(24) $(X \sqcup Y) \setminus Z = (X \setminus Z) \sqcup (Y \setminus Z)$.

[2]The proposition (19) has been removed.

4. Distributive lower bounded lattice

In the sequel L will denote a lower bound lattice and X, Y, Z will denote elements of the carrier of L.

The following propositions are true:

(25) If $X \sqsubseteq \bot_L$, then $X = \bot_L$.

(26) If $X \sqsubseteq Y$ and $X \sqsubseteq Z$ and $Y \sqcap Z = \bot_L$, then $X = \bot_L$.

(27) $X \sqcup Y = \bot_L$ iff $X = \bot_L$ and $Y = \bot_L$.

(28) If $X \sqsubseteq Y$ and $Y \sqcap Z = \bot_L$, then $X \sqcap Z = \bot_L$.

(29) $\bot_L \setminus X = \bot_L$.

(30) If X meets Y and $Y \sqsubseteq Z$, then X meets Z.

(31) If X meets $Y \sqcap Z$, then X meets Y and X meets Z.

(32) If X meets $Y \setminus Z$, then X meets Y.

(33) X misses $\bot_L$.

(34) If X misses Z and $Y \sqsubseteq Z$, then X misses Y.

(35) If X misses Y or X misses Z, then X misses $Y \sqcap Z$.

(36) If $X \sqsubseteq Y$ and $X \sqsubseteq Z$ and Y misses Z, then $X = \bot_L$.

(37) If X misses Y, then $Z \sqcap X$ misses $Z \sqcap Y$ and $X \sqcap Z$ misses $Y \sqcap Z$.

5. Boolean lattice

We follow a convention: L will be a Boolean lattice and X, Y, Z, V will be elements of the carrier of L.

Next we state a number of propositions:

(38) If $X \setminus Y \sqsubseteq Z$, then $X \sqsubseteq Y \sqcup Z$.

(39) If $X \sqsubseteq Y$, then $Z \setminus Y \sqsubseteq Z \setminus X$.

(40) If $X \sqsubseteq Y$ and $Z \sqsubseteq V$, then $X \setminus V \sqsubseteq Y \setminus Z$.

(41) If $X \sqsubseteq Y \sqcup Z$, then $X \setminus Y \sqsubseteq Z$ and $X \setminus Z \sqsubseteq Y$.

(42) $X^{\mathrm{c}} \sqsubseteq (X \sqcap Y)^{\mathrm{c}}$ and $Y^{\mathrm{c}} \sqsubseteq (X \sqcap Y)^{\mathrm{c}}$.

(43) $(X \sqcup Y)^{\mathrm{c}} \sqsubseteq X^{\mathrm{c}}$ and $(X \sqcup Y)^{\mathrm{c}} \sqsubseteq Y^{\mathrm{c}}$.

(44) If $X \sqsubseteq Y \setminus X$, then $X = \bot_L$.

(45) If $X \sqsubseteq Y$, then $Y = X \sqcup (Y \setminus X)$ and $Y = (Y \setminus X) \sqcup X$.

(46) $X \setminus Y = \bot_L$ iff $X \sqsubseteq Y$.

(47) If $X \sqsubseteq Y \sqcup Z$ and $X \sqcap Z = \bot_L$, then $X \sqsubseteq Y$.

(48) $X \sqcup Y = (X \setminus Y) \sqcup Y$.

(49) $X \setminus (X \sqcup Y) = \bot_L$ and $X \setminus (Y \sqcup X) = \bot_L$.

(50) $X \setminus X \sqcap Y = X \setminus Y$ and $X \setminus Y \sqcap X = X \setminus Y$.

(51) $(X \setminus Y) \sqcap Y = \bot_L$ and $Y \sqcap (X \setminus Y) = \bot_L$.

(52) $X \sqcup (Y \setminus X) = X \sqcup Y$ and $(Y \setminus X) \sqcup X = Y \sqcup X$.

(53) $X \sqcap Y \sqcup (X \setminus Y) = X$ and $(X \setminus Y) \sqcup X \sqcap Y = X$.

(54) $X \setminus (Y \setminus Z) = (X \setminus Y) \sqcup X \sqcap Z$.

(55) $X \setminus (X \setminus Y) = X \sqcap Y$.

(56) $(X \sqcup Y) \setminus Y = X \setminus Y$.

(57) $X \sqcap Y = \bot_L$ iff $X \setminus Y = X$.

(58) $X \setminus (Y \sqcup Z) = (X \setminus Y) \sqcap (X \setminus Z)$.

(59) $X \setminus Y \sqcap Z = (X \setminus Y) \sqcup (X \setminus Z)$.

(60) $X \sqcap (Y \setminus Z) = X \sqcap Y \setminus X \sqcap Z$ and $(Y \setminus Z) \sqcap X = Y \sqcap X \setminus Z \sqcap X$.

(61) $(X \sqcup Y) \setminus X \sqcap Y = (X \setminus Y) \sqcup (Y \setminus X)$.

(62) $X \setminus Y \setminus Z = X \setminus (Y \sqcup Z)$.

(63) If $X \setminus Y = Y \setminus X$, then $X = Y$.

(64) $(\bot_L)^{\rm c} = \top_L$.

(65) $(\top_L)^{\rm c} = \bot_L$.

(66) $X \setminus X = \bot_L$.

(67) $X \setminus \bot_L = X$.

(68) $(X \setminus Y)^{\rm c} = X^{\rm c} \sqcup Y$.

(69) X meets $Y \sqcup Z$ iff X meets Y or X meets Z.

(70) $X \sqcap Y$ misses $X \setminus Y$.

(71) X misses $Y \sqcup Z$ iff X misses Y and X misses Z.

(72) $X \setminus Y$ misses Y.

(73) If X misses Y, then $(X \sqcup Y) \setminus Y = X$ and $(X \sqcup Y) \setminus X = Y$.

(74) If $X^{\rm c} \sqcup Y^{\rm c} = X \sqcup Y$ and X misses $X^{\rm c}$ and Y misses $Y^{\rm c}$, then $X = Y^{\rm c}$ and $Y = X^{\rm c}$.

(75) If $X^{\rm c} \sqcup Y^{\rm c} = X \sqcup Y$ and Y misses $X^{\rm c}$ and X misses $Y^{\rm c}$, then $X = X^{\rm c}$ and $Y = Y^{\rm c}$.

(76) $X \dot{-} \bot_L = X$ and $\bot_L \dot{-} X = X$.

(77) $X \dot{-} X = \bot_L$.

(78) $X \sqcap Y$ misses $X \dot{-} Y$.

(79) $X \sqcup Y = X \dot{-} (Y \setminus X)$.

(80) $X \dot{-} X \sqcap Y = X \setminus Y$.

(81) $X \sqcup Y = (X \dot{-} Y) \sqcup X \sqcap Y$.

(82) $X \dot{-} Y \dot{-} X \sqcap Y = X \sqcup Y$.

(83) $X \dot{-} Y \dot{-} (X \sqcup Y) = X \sqcap Y$.

(84) $X \dot{-} Y = (X \sqcup Y) \setminus X \sqcap Y$.

(85) $(X \dot{-} Y) \setminus Z = (X \setminus (Y \sqcup Z)) \sqcup (Y \setminus (X \sqcup Z))$.

(86) $X \setminus (Y \dot{-} Z) = (X \setminus (Y \sqcup Z)) \sqcup X \sqcap Y \sqcap Z$.

(87) $(X \dot{-} Y) \dot{-} Z = X \dot{-} (Y \dot{-} Z)$.

(88) $(X \dot{-} Y)^{\rm c} = X \sqcap Y \sqcup X^{\rm c} \sqcap Y^{\rm c}$.

References

[1] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

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Many Sorted Algebras

Andrzej Trybulec
Warsaw University
Białystok

Summary. The basic purpose of the paper is to prepare preliminaries of the theory of many sorted algebras. The concept of the signature of a many sorted algebra is introduced as well as the concept of many sorted algebra itself. Some auxiliary related notions are defined. The correspondence between (1 sorted) universal algebras [9] and many sorted algebras with one sort only is described by introducing two functors mapping one into the other. The construction is done this way that the composition of both functors is the identity on universal algebras.

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The articles [12], [14], [5], [6], [2], [10], [7], [4], [1], [11], [13], [3], [8], and [9] provide the notation and terminology for this paper.

1. Preliminaries

In this paper i, j are arbitrary and I is a set.

Next we state the proposition

(1) It is not true that there exists a non-empty many sorted set M of I such that $\emptyset \in \operatorname{rng} M$.

In this article we present several logical schemes. The scheme *MSSEx* deals with a set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a many sorted set f of $\mathcal{A}$ such that for every i such that $i \in \mathcal{A}$ holds $\mathcal{P}[i, f(i)]$

provided the following condition is met:

- For every i such that $i \in \mathcal{A}$ there exists j such that $\mathcal{P}[i, j]$.

The scheme *MSSLambda* concerns a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

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There exists a many sorted set f of $\mathcal{A}$ such that for every i such that $i \in \mathcal{A}$ holds $f(i) = \mathcal{F}(i)$

for all values of the parameters.

Let I be a set and let M be a many sorted set of I. A component of M is an element of $\operatorname{rng} M$.

Next we state two propositions:

(2) Let I be a non empty set, and let M be a many sorted set of I, and let A be a component of M. Then there exists i such that $i \in I$ and $A = M(i)$.

(3) For every many sorted set M of I and for every i such that $i \in I$ holds $M(i)$ is a component of M.

Let us consider I and let B be a many sorted set of I. A many sorted set of I is said to be an element of B if:

(Def.1) For every i such that $i \in I$ holds $\mathrm{it}(i)$ is an element of $B(i)$.

2. Auxiliary functors

Let us consider I, let A be a many sorted set of I, and let B be a many sorted set of I. A many sorted set of I is called a many sorted function from A into B if:

(Def.2) For every i such that $i \in I$ holds $\mathrm{it}(i)$ is a function from $A(i)$ into $B(i)$.

Let us consider I, let A be a many sorted set of I, and let B be a many sorted set of I. Note that every many sorted function from A into B is function yielding.

Let I be a set and let M be a many sorted set of I. The functor $M^{\#}$ yielding a many sorted set of I^* is defined by:

(Def.3) For every element i of I^* holds $M^{\#}(i) = \prod(M \cdot i)$.

Let I be a set and let M be a non-empty many sorted set of I. Note that $M^{\#}$ is non-empty.

Let us consider I, let J be a non empty set, let O be a function from I into J, and let F be a many sorted set of J. Then $F \cdot O$ is a many sorted set of I.

Let us consider I, let J be a non empty set, let O be a function from I into J, and let F be a non-empty many sorted set of J. Then $F \cdot O$ is a non-empty many sorted set of I.

Let a be arbitrary. The functor $\square \longmapsto a$ yields a function from $\mathbb{N}$ into $\{a\}^*$ and is defined as follows:

(Def.4) For every natural number n holds $(\square \longmapsto a)(n) = n \mapsto a$.

In the sequel D denotes a non empty set and n denotes a natural number.

The following propositions are true:

(4) For arbitrary a, b holds $(\{a\} \longmapsto b) \cdot (n \mapsto a) = n \mapsto b$.

(5) For arbitrary a and for every many sorted set M of $\{a\}$ such that $M = \{a\} \longmapsto D$ holds $(M^{\#} \cdot (\square \longmapsto a))(n) = D^{\operatorname{Seg} n}$.

Let us consider I, i. Then $I \longmapsto i$ is a function from I into $\{i\}$.

Let C be a set, let A, B be non empty sets, let F be a partial function from C to A, and let G be a function from A into B. Then $G \cdot F$ is a function from dom F into B.

3. Many sorted signatures

We introduce many sorted signatures which are extensions of 1-sorted structure and are systems

⟨ a carrier, operation symbols, an arity, a result sort ⟩,

where the carrier is a set, the operation symbols constitute a set, the arity is a function from the operation symbols into the carrier*, and the result sort is a function from the operation symbols into the carrier.

A many sorted signature is void if:

(Def.5) The operation symbols of it $= \emptyset$.

One can verify that there exists a many sorted signature which is void strict and non empty and there exists a many sorted signature which is non void strict and non empty.

In the sequel S is a non empty many sorted signature.

Let us consider S. A sort symbol of S is an element of the carrier of S. An operation symbol of S is an element of the operation symbols of S.

Let S be a non void non empty many sorted signature and let o be an operation symbol of S. The functor Arity(o) yields an element of (the carrier of S)* and is defined as follows:

(Def.6) Arity(o) = (the arity of S)(o).

The result sort of o yielding an element of the carrier of S is defined by:

(Def.7) The result sort of o = (the result sort of S)(o).

4. Many sorted algebras

Let S be a 1-sorted structure. We consider many-sorted structures over S as systems

⟨ sorts ⟩,

where the sorts constitute a many sorted set of the carrier of S.

Let us consider S. We consider algebras over S as extensions of many-sorted structure over S as systems

⟨ sorts, a characteristics ⟩,

where the sorts constitute a many sorted set of the carrier of S and the characteristics is a many sorted function from the sorts$^{\#} \cdot$ (the arity of S) into (the sorts) $\cdot$(the result sort of S).

Let us consider S and let A be an algebra over S. We say that A is non-empty if and only if:

(Def.8) The sorts of A is non-empty.

Let us consider S. Observe that there exists an algebra over S which is strict and non-empty.

Let us consider S and let A be a non-empty algebra over S. One can verify that the sorts of A is non-empty.

Let us consider S and let A be a non-empty algebra over S. One can check that every component of the sorts of A is non empty and every component of the sorts of $A^{\#}$ is non empty.

Let S be a non void non empty many sorted signature, let o be an operation symbol of S, and let A be an algebra over S. The functor $\mathrm{Args}(o, A)$ yielding a component of (the sorts of $A)^{\#}$ is defined by:

(Def.9) $\mathrm{Args}(o, A) = ((\text{the sorts of } A)^{\#} \cdot (\text{the arity of } S))(o)$.

The functor $\mathrm{Result}(o, A)$ yields a component of the sorts of A and is defined as follows:

(Def.10) $\mathrm{Result}(o, A) = ((\text{the sorts of } A) \cdot (\text{the result sort of } S))(o)$.

Let S be a non void non empty many sorted signature, let o be an operation symbol of S, and let A be an algebra over S. The functor $\mathrm{Den}(o, A)$ yielding a function from $\mathrm{Args}(o, A)$ into $\mathrm{Result}(o, A)$ is defined as follows:

(Def.11) $\mathrm{Den}(o, A) = (\text{the characteristics of } A)(o)$.

The following proposition is true

(6) Let S be a non void non empty many sorted signature, and let o be an operation symbol of S, and let A be a non-empty algebra over S. Then $\mathrm{Den}(o, A)$ is non empty.

5. Universal algebras as many sorted

We now state two propositions:

(8)[1] For every homogeneous quasi total non empty partial function h from D^* to D holds $\mathrm{dom}\, h = D^{\mathrm{Seg\, arity}\, h}$.

(9) For every universal algebra A holds signature A is non empty.

6. Universal algebras for many sorted algebras with one sort

Let A be a universal algebra. Then signature A is a finite sequence of elements of $\mathbb{N}$.

A many sorted signature is segmental if:

(Def.12) There exists n such that the operation symbols of it $= \mathrm{Seg}\, n$.

The following proposition is true

[1] The proposition (7) has been removed.

(10) Let S be a non empty many sorted signature. Suppose S is trivial. Let A be an algebra over S and let c_1, c_2 be components of the sorts of A. Then $c_1 = c_2$.

Let us mention that there exists a many sorted signature which is segmental trivial non void strict and non empty.

Let A be a universal algebra. The functor MSSign(A) yields a non void strict segmental trivial many sorted signature and is defined by:

(Def.13) $\mathrm{MSSign}(A) = \langle\{0\}, \mathrm{dom\,signature}\,A, (\square \longmapsto 0) \cdot \mathrm{signature}\,A, \mathrm{dom\,signature}\,A \longmapsto 0\rangle$.

Let A be a universal algebra. One can check that MSSign(A) is non empty.

Let A be a universal algebra. The functor MSSorts(A) yields a non-empty many sorted set of the carrier of MSSign(A) and is defined as follows:

(Def.14) $\mathrm{MSSorts}(A) = \{0\} \longmapsto$ the carrier of A.

Let A be a universal algebra. The functor MSCharact(A) yields a many sorted function from $(\mathrm{MSSorts}(A))^{\#} \cdot$(the arity of MSSign($A$)) into $\mathrm{MSSorts}(A) \cdot$ (the result sort of MSSign(A)) and is defined by:

(Def.15) MSCharact(A) = the characteristic of A.

Let A be a universal algebra. The functor MSAlg(A) yielding a strict algebra over MSSign(A) is defined by:

(Def.16) $\mathrm{MSAlg}(A) = \langle\mathrm{MSSorts}(A), \mathrm{MSCharact}(A)\rangle$.

Let A be a universal algebra. Note that MSAlg(A) is non-empty.

Let M_1 be a trivial non empty many sorted signature and let A be an algebra over M_1. The sort of A yielding a set is defined as follows:

(Def.17) There exists a component c of the sorts of A such that the sort of $A = c$.

Let M_1 be a trivial non empty many sorted signature and let A be a non-empty algebra over M_1. Observe that the sort of A is non empty.

We now state four propositions:

(11) Let M_1 be a segmental trivial non void non empty many sorted signature, and let i be an operation symbol of M_1, and let A be a non-empty algebra over M_1. Then $\mathrm{Args}(i, A) = (\text{the sort of } A)^{\mathrm{len\,Arity}(i)}$.

(12) For every non empty set A and for every n holds $A^n \subseteq A^*$.

(13) Let M_1 be a segmental trivial non void non empty many sorted signature, and let i be an operation symbol of M_1, and let A be a non-empty algebra over M_1. Then $\mathrm{Args}(i, A) \subseteq (\text{the sort of } A)^*$.

(14) Let M_1 be a segmental trivial non void non empty many sorted signature and let A be a non-empty algebra over M_1. Then the characteristics of A is a finite sequence of elements of $(\text{the sort of } A)^* \dot{\rightarrow} \text{the sort of } A$.

Let M_1 be a segmental trivial non void non empty many sorted signature and let A be a non-empty algebra over M_1. The functor charact(A) yielding a finite sequence of operational functions of the sort of A is defined by:

(Def.18) charact(A) = the characteristics of A.

In the sequel M_1 will denote a segmental trivial non void non empty many sorted signature and A will denote a non-empty algebra over M_1.

Let us consider M_1, A. The functor $\mathrm{Alg}_{\mathbf{1}}(A)$ yields a non-empty strict universal algebra and is defined as follows:

(Def.19) $\mathrm{Alg}_{\mathbf{1}}(A) = \langle$the sort of A, $\mathrm{charact}(A)\rangle$.

We now state the proposition

(15) For every strict universal algebra A holds $A = \mathrm{Alg}_{\mathbf{1}}(\mathrm{MSAlg}(A))$.

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References

[1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.
[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[8] Zbigniew Karno. Maximal discrete subspaces of almost discrete topological spaces. *Formalized Mathematics*, 4(**1**):125–135, 1993.
[9] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Formalized Mathematics*, 3(**2**):251–253, 1992.
[10] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[11] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[13] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[14] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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On the Group of Inner Automorphisms

Artur Korniłowicz
Warsaw University
Białystok

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The notation and terminology used in this paper are introduced in the following articles: [6], [2], [3], [1], [5], [11], [4], [9], [10], [7], [8], and [12].

For simplicity we adopt the following rules: G denotes a strict group, H denotes a subgroup of G, a, b, x denote elements of G, and h denotes a homomorphism from G to G.

One can prove the following proposition

(1) For all a, b such that b is an element of H holds $b^a \in H$ iff H is normal.

Let us consider G. One can verify that $\mathrm{Z}(G)$ is normal.

Let us consider G. The functor $\mathrm{Aut}(G)$ yields a non empty set of functions from the carrier of G to the carrier of G and is defined as follows:

(Def.1) Every element of $\mathrm{Aut}(G)$ is a homomorphism from G to G and for every h holds $h \in \mathrm{Aut}(G)$ iff h is one-to-one and an epimorphism.

We now state several propositions:

(2) For every h holds $h \in \mathrm{Aut}(G)$ iff h is one-to-one and an epimorphism.

(3) $\mathrm{Aut}(G) \subseteq (\text{the carrier of } G)^{\text{the carrier of } G}$.

(4) $\mathrm{id}_{(\text{the carrier of } G)}$ is an element of $\mathrm{Aut}(G)$.

(5) For every h holds $h \in \mathrm{Aut}(G)$ iff h is an isomorphism.

(6) For every element f of $\mathrm{Aut}(G)$ holds f^{-1} is a homomorphism from G to G.

(7) For every element f of $\mathrm{Aut}(G)$ holds f^{-1} is an element of $\mathrm{Aut}(G)$.

(8) For all elements f_1, f_2 of $\mathrm{Aut}(G)$ holds $f_1 \cdot f_2$ is an element of $\mathrm{Aut}(G)$.

Let us consider G. The functor $\mathrm{AutComp}(G)$ yielding a binary operation on $\mathrm{Aut}(G)$ is defined as follows:

(Def.2) For all elements x, y of $\mathrm{Aut}(G)$ holds $(\mathrm{AutComp}(G))(x, y) = x \cdot y$.

Let us consider G. The functor $\mathrm{AutGroup}(G)$ yields a strict group and is defined by:

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(Def.3) $\mathrm{AutGroup}(G) = \langle \mathrm{Aut}(G), \mathrm{AutComp}(G) \rangle$.

The following three propositions are true:

(9) For all elements x, y of $\mathrm{AutGroup}(G)$ and for all elements f, g of $\mathrm{Aut}(G)$ such that $x = f$ and $y = g$ holds $x \cdot y = f \cdot g$.

(10) $\mathrm{id}_{(\text{the carrier of } G)} = 1_{\mathrm{AutGroup}(G)}$.

(11) For every element f of $\mathrm{Aut}(G)$ and for every element g of $\mathrm{AutGroup}(G)$ such that $f = g$ holds $f^{-1} = g^{-1}$.

Let us consider G. The functor $\mathrm{InnAut}(G)$ yields a non empty set of functions from the carrier of G to the carrier of G and is defined by the condition (Def.4).

(Def.4) Let f be an element of (the carrier of $G)^{\text{the carrier of } G}$. Then $f \in \mathrm{InnAut}(G)$ if and only if there exists a such that for every x holds $f(x) = x^a$.

Next we state several propositions:

(12) $\mathrm{InnAut}(G) \subseteq (\text{the carrier of } G)^{\text{the carrier of } G}$.

(13) Every element of $\mathrm{InnAut}(G)$ is an element of $\mathrm{Aut}(G)$.

(14) $\mathrm{InnAut}(G) \subseteq \mathrm{Aut}(G)$.

(15) For all elements f, g of $\mathrm{InnAut}(G)$ holds $(\mathrm{AutComp}(G))(f, g) = f \cdot g$.

(16) $\mathrm{id}_{(\text{the carrier of } G)}$ is an element of $\mathrm{InnAut}(G)$.

(17) For every element f of $\mathrm{InnAut}(G)$ holds f^{-1} is an element of $\mathrm{InnAut}(G)$.

(18) For all elements f, g of $\mathrm{InnAut}(G)$ holds $f \cdot g$ is an element of $\mathrm{InnAut}(G)$.

Let us consider G. The functor $\mathrm{InnAutGroup}(G)$ yields a normal strict subgroup of $\mathrm{AutGroup}(G)$ and is defined by:

(Def.5) The carrier of $\mathrm{InnAutGroup}(G) = \mathrm{InnAut}(G)$.

Next we state three propositions:

(20)[1] For all elements x, y of $\mathrm{InnAutGroup}(G)$ and for all elements f, g of $\mathrm{InnAut}(G)$ such that $x = f$ and $y = g$ holds $x \cdot y = f \cdot g$.

(21) $\mathrm{id}_{(\text{the carrier of } G)} = 1_{\mathrm{InnAutGroup}(G)}$.

(22) For every element f of $\mathrm{InnAut}(G)$ and for every element g of $\mathrm{InnAutGroup}(G)$ such that $f = g$ holds $f^{-1} = g^{-1}$.

Let us consider G, b. The functor $\mathrm{Conjugate}(b)$ yields an element of $\mathrm{InnAut}(G)$ and is defined by:

(Def.6) For every a holds $(\mathrm{Conjugate}(b))(a) = a^b$.

The following propositions are true:

(23) For all a, b holds $\mathrm{Conjugate}(a \cdot b) = \mathrm{Conjugate}(b) \cdot \mathrm{Conjugate}(a)$.

(24) $\mathrm{Conjugate}(1_G) = \mathrm{id}_{(\text{the carrier of } G)}$.

(25) For every a holds $(\mathrm{Conjugate}(1_G))(a) = a$.

(26) For every a holds $\mathrm{Conjugate}(a) \cdot \mathrm{Conjugate}(a^{-1}) = \mathrm{Conjugate}(1_G)$.

(27) For every a holds $\mathrm{Conjugate}(a^{-1}) \cdot \mathrm{Conjugate}(a) = \mathrm{Conjugate}(1_G)$.

(28) For every a holds $\mathrm{Conjugate}(a^{-1}) = (\mathrm{Conjugate}(a))^{-1}$.

[1]The proposition (19) has been removed.

(29) For every a holds $\mathrm{Conjugate}(a) \cdot \mathrm{Conjugate}(1_G) = \mathrm{Conjugate}(a)$ and $\mathrm{Conjugate}(1_G) \cdot \mathrm{Conjugate}(a) = \mathrm{Conjugate}(a)$.

(30) For every element f of $\mathrm{InnAut}(G)$ holds $f \cdot \mathrm{Conjugate}(1_G) = f$ and $\mathrm{Conjugate}(1_G) \cdot f = f$.

(31) For every G holds $\mathrm{InnAutGroup}(G)$ and ${}^{G}/_{\mathrm{Z}(G)}$ are isomorphic.

(32) For every G such that G is a commutative group and for every element f of $\mathrm{InnAutGroup}(G)$ holds $f = 1_{\mathrm{InnAutGroup}(G)}$.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[5] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[7] Wojciech A. Trybulec. Classes of conjugation. Normal subgroups. *Formalized Mathematics*, 1(**5**):955–962, 1990.
[8] Wojciech A. Trybulec. Commutator and center of a group. *Formalized Mathematics*, 2(**4**):461–466, 1991.
[9] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[10] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Formalized Mathematics*, 1(**5**):855–864, 1990.
[11] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[12] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(**4**):573–578, 1991.

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Subalgebras of Many Sorted Algebra. Lattice of Subalgebras

Ewa Burakowska
Warsaw University
Białystok

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The articles [12], [13], [5], [6], [2], [8], [9], [7], [4], [14], [3], [1], [11], and [10] provide the notation and terminology for this paper.

1. Auxilary Facts about Many Sorted Sets

In this paper x will be arbitrary.

The scheme *LambdaB* concerns a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a function f such that $\operatorname{dom} f = \mathcal{A}$ and for every element d of $\mathcal{A}$ holds $f(d) = \mathcal{F}(d)$

for all values of the parameters.

Let I be a set, let X be a many sorted set of I, and let Y be a non-empty many sorted set of I. Observe that $X \cup Y$ is non-empty and $Y \cup X$ is non-empty.

Next we state two propositions:

(1) Let I be a set, and let X be a many sorted set of I, and let Y be a non-empty many sorted set of I. Then $X \cup Y$ is non-empty and $Y \cup X$ is non-empty.

(2) For every non empty set I and for all many sorted sets X, Y of I and for every element i of I^* holds $\prod((X \cap Y) \cdot i) = \prod(X \cdot i) \cap \prod(Y \cdot i)$.

Let I be a set and let M be a many sorted set of I. A many sorted set of I is said to be a many sorted subset of M if:

(Def.1) It $\subseteq M$.

Let I be a set and let M be a non-empty many sorted set of I. Observe that there exists a many sorted subset of M which is non-empty.

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2. Constants of a Many Sorted Algebra

We follow the rules: S will denote a non void non empty many sorted signature, o will denote an operation symbol of S, and U_0, U_1, U_2 will denote algebras over S.

Let S be a non empty many sorted signature and let U_0 be an algebra over S. A subset of U_0 is a many sorted subset of the sorts of U_0.

Let S be a non empty many sorted signature. A sort symbol of S has constants if:

(Def.2) There exists an operation symbol o of S such that (the arity of S)$(o) = \varepsilon$ and (the result sort of S)$(o) =$ it.

A non empty many sorted signature has constant operations if:

(Def.3) Every sort symbol of it has constants.

Let A be a non empty set, let B be a set, let a be a function from B into A^*, and let r be a function from B into A. Note that $\langle A, B, a, r\rangle$ is non empty.

Let us observe that there exists a non empty many sorted signature which is non void and strict and has constant operations.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let s be a sort symbol of S. The functor Constants(U_0, s) yielding a subset of (the sorts of U_0)(s) is defined by:

(Def.4) (i) There exists a non empty set A such that $A =$ (the sorts of U_0)(s) and Constants$(U_0, s) = \{a : a$ ranges over elements of A, $\bigvee_o$ (the arity of S)$(o) = \varepsilon \ \wedge$ (the result sort of S)$(o) = s \ \wedge \ a \in \operatorname{rng} \operatorname{Den}(o, U_0)\}$ if (the sorts of U_0)$(s) \neq \emptyset$,

(ii) Constants$(U_0, s) = \emptyset$, otherwise.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. The functor Constants(U_0) yielding a subset of U_0 is defined as follows:

(Def.5) For every sort symbol s of S holds (Constants(U_0))$(s) =$ Constants(U_0, s).

Let S be a non void non empty many sorted signature with constant operations, let U_0 be a non-empty algebra over S, and let s be a sort symbol of S. One can verify that Constants(U_0, s) is non empty.

Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. One can verify that Constants(U_0) is non-empty.

3. Subalgebras of a Many Sorted Algebra

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, let o be an operation symbol of S, and let A be a subset of U_0. We say that A is closed on o if and only if:

(Def.6) $\mathrm{rng}(\mathrm{Den}(o, U_0) \restriction (A^{\#} \cdot (\text{the arity of } S))(o)) \subseteq (A \cdot (\text{the result sort of } S))(o)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. We say that A is operations closed if and only if:

(Def.7) For every operation symbol o of S holds A is closed on o.

One can prove the following proposition

(3) Let S be a non void non empty many sorted signature, and let o be an operation symbol of S, and let U_0 be an algebra over S, and let B_0, B_1 be subsets of U_0. If $B_0 \subseteq B_1$, then $({B_0}^{\#} \cdot (\text{the arity of } S))(o) \subseteq ({B_1}^{\#} \cdot (\text{the arity of } S))(o)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, let o be an operation symbol of S, and let A be a subset of U_0. Let us assume that A is closed on o. The functor o_A yielding a function from $(A^{\#} \cdot (\text{the arity of } S))(o)$ into $(A \cdot (\text{the result sort of } S))(o)$ is defined as follows:

(Def.8) $o_A = \mathrm{Den}(o, U_0) \restriction (A^{\#} \cdot (\text{the arity of } S))(o)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. The functor $\mathrm{Opers}(U_0, A)$ yielding a many sorted function from $A^{\#} \cdot (\text{the arity of } S)$ into $A \cdot (\text{the result sort of } S)$ is defined by:

(Def.9) For every operation symbol o of S holds $(\mathrm{Opers}(U_0, A))(o) = o_A$.

Next we state two propositions:

(4) Let U_0 be an algebra over S and let B be a subset of U_0. Suppose B = the sorts of U_0. Then B is operations closed and for every o holds $o_B = \mathrm{Den}(o, U_0)$.

(5) For every subset B of U_0 such that B = the sorts of U_0 holds $\mathrm{Opers}(U_0, B)$ = the characteristics of U_0.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. An algebra over S is called a subalgebra of U_0 if it satisfies the conditions (Def.10).

(Def.10) (i) The sorts of it is a subset of U_0, and

(ii) for every subset B of U_0 such that B = the sorts of it holds B is operations closed and the characteristics of it $= \mathrm{Opers}(U_0, B)$.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. One can check that there exists a subalgebra of U_0 which is strict.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. Observe that there exists a subalgebra of U_0 which is non-empty and strict.

One can prove the following propositions:

(6) U_0 is a subalgebra of U_0.

(7) If U_0 is a subalgebra of U_1 and U_1 is a subalgebra of U_2, then U_0 is a subalgebra of U_2.

(8) If U_1 is a strict subalgebra of U_2 and U_2 is a strict subalgebra of U_1, then $U_1 = U_2$.

(9) For all subalgebras U_1, U_2 of U_0 such that the sorts of $U_1 \subseteq$ the sorts of U_2 holds U_1 is a subalgebra of U_2.

(10) For all strict subalgebras U_1, U_2 of U_0 such that the sorts of $U_1 =$ the sorts of U_2 holds $U_1 = U_2$.

(11) Let S be a non void non empty many sorted signature, and let U_0 be an algebra over S, and let U_1 be a subalgebra of U_0. Then Constants(U_0) is a subset of U_1.

(12) Let S be a non void non empty many sorted signature with constant operations, and let U_0 be a non-empty algebra over S, and let U_1 be a non-empty subalgebra of U_0. Then Constants(U_0) is a non-empty subset of U_1.

(13) Let S be a non void non empty many sorted signature with constant operations, and let U_0 be a non-empty algebra over S, and let U_1, U_2 be non-empty subalgebras of U_0. Then (the sorts of U_1) $\cap$ (the sorts of U_2) is non-empty.

4. Many Sorted Subsets of Many Sorted Algebra

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. The functor SubSorts(A) yielding a non empty set is defined by the condition (Def.11).

(Def.11) Let x be arbitrary. Then $x \in$ SubSorts(A) if and only if the following conditions are satisfied:

(i) $x \in (2^{\bigcup (\text{the sorts of } U_0)})^{\text{the carrier of } S}$,

(ii) x is a subset of U_0, and

(iii) for every subset B of U_0 such that $B = x$ holds B is operations closed and Constants(U_0) $\subseteq B$ and $A \subseteq B$.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. The functor SubSorts(U_0) yields a non empty set and is defined by the condition (Def.12).

(Def.12) Let x be arbitrary. Then $x \in$ SubSorts(U_0) if and only if the following conditions are satisfied:

(i) $x \in (2^{\bigcup (\text{the sorts of } U_0)})^{\text{the carrier of } S}$,

(ii) x is a subset of U_0, and

(iii) for every subset B of U_0 such that $B = x$ holds B is operations closed.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let e be an element of SubSorts(U_0). The functor ${}^{@}e$ yielding a subset of U_0 is defined as follows:

(Def.13) $^{@}e = e$.

Next we state two propositions:

(14) For all subsets A, B of U_0 holds $B \in \mathrm{SubSorts}(A)$ iff B is operations closed and $\mathrm{Constants}(U_0) \subseteq B$ and $A \subseteq B$.

(15) For every subset B of U_0 holds $B \in \mathrm{SubSorts}(U_0)$ iff B is operations closed.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, let A be a subset of U_0, and let s be a sort symbol of S. The functor $\mathrm{SubSort}(A, s)$ yields a non empty set and is defined as follows:

(Def.14) For arbitrary x holds $x \in \mathrm{SubSort}(A, s)$ iff there exists a subset B of U_0 such that $B \in \mathrm{SubSorts}(A)$ and $x = B(s)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. The functor $\mathrm{MSSubSort}(A)$ yields a subset of U_0 and is defined as follows:

(Def.15) For every sort symbol s of S holds $(\mathrm{MSSubSort}(A))(s) = \bigcap \mathrm{SubSort}(A, s)$.

We now state several propositions:

(16) For every subset A of U_0 holds $\mathrm{Constants}(U_0) \cup A \subseteq \mathrm{MSSubSort}(A)$.

(17) For every subset A of U_0 such that $\mathrm{Constants}(U_0) \cup A$ is non-empty holds $\mathrm{MSSubSort}(A)$ is non-empty.

(18) Let A be a subset of U_0 and let B be a subset of U_0. If $B \in \mathrm{SubSorts}(A)$, then $((\mathrm{MSSubSort}(A))^{\#} \cdot (\text{the arity of } S))(o) \subseteq (B^{\#} \cdot (\text{the arity of } S))(o)$.

(19) Let A be a subset of U_0 and let B be a subset of U_0. Suppose $B \in \mathrm{SubSorts}(A)$. Then $\mathrm{rng}(\mathrm{Den}(o, U_0) \restriction ((\mathrm{MSSubSort}(A))^{\#} \cdot (\text{the arity of } S))(o)) \subseteq (B \cdot (\text{the result sort of } S))(o)$.

(20) For every subset A of U_0 holds $\mathrm{rng}(\mathrm{Den}(o, U_0) \restriction ((\mathrm{MSSubSort}(A))^{\#} \cdot (\text{the arity of } S))(o)) \subseteq (\mathrm{MSSubSort}(A) \cdot (\text{the result sort of } S))(o)$.

(21) For every subset A of U_0 holds $\mathrm{MSSubSort}(A)$ is operations closed and $A \subseteq \mathrm{MSSubSort}(A)$.

5. Operations on Many Sorted Algebra and its Subalgebras

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. Let us assume that A is operations closed. The functor $U_0 \restriction A$ yields a strict subalgebra of U_0 and is defined as follows:

(Def.16) $U_0 \restriction A = \langle A, (\mathrm{Opers}(U_0, A)$ **qua** many sorted function from $A^{\#} \cdot (\text{the arity of } S)$ into $A \cdot (\text{the result sort of } S))\rangle$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let U_1, U_2 be subalgebras of U_0. The functor $U_1 \cap U_2$ yielding a strict subalgebra of U_0 is defined by the conditions (Def.17).

(Def.17) (i) The sorts of $U_1 \cap U_2$ = (the sorts of U_1) $\cap$ (the sorts of U_2), and

(ii) for every subset B of U_0 such that B = the sorts of $U_1 \cap U_2$ holds B is operations closed and the characteristics of $U_1 \cap U_2 = \mathrm{Opers}(U_0, B)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. The functor $\mathrm{Gen}(A)$ yields a strict subalgebra of U_0 and is defined by the conditions (Def.18).

(Def.18) (i) A is a subset of $\mathrm{Gen}(A)$, and

(ii) for every subalgebra U_1 of U_0 such that A is a subset of U_1 holds $\mathrm{Gen}(A)$ is a subalgebra of U_1.

Let S be a non void non empty many sorted signature, let U_0 be a non-empty algebra over S, and let A be a non-empty subset of U_0. Observe that $\mathrm{Gen}(A)$ is non-empty.

We now state three propositions:

(22) Let S be a non void non empty many sorted signature, and let U_0 be a strict algebra over S, and let B be a subset of U_0. If B = the sorts of U_0, then $\mathrm{Gen}(B) = U_0$.

(23) Let S be a non void non empty many sorted signature, and let U_0 be an algebra over S, and let U_1 be a strict subalgebra of U_0, and let B be a subset of U_0. If B = the sorts of U_1, then $\mathrm{Gen}(B) = U_1$.

(24) Let S be a non void non empty many sorted signature with constant operations, and let U_0 be a non-empty algebra over S, and let U_1 be a subalgebra of U_0. Then $\mathrm{Gen}(\mathrm{Constants}(U_0)) \cap U_1 = \mathrm{Gen}(\mathrm{Constants}(U_0))$.

Let S be a non void non empty many sorted signature, let U_0 be a non-empty algebra over S, and let U_1, U_2 be subalgebras of U_0. The functor $U_1 \bigsqcup U_2$ yielding a strict subalgebra of U_0 is defined as follows:

(Def.19) For every subset A of U_0 such that A = (the sorts of U_1) $\cup$ (the sorts of U_2) holds $U_1 \bigsqcup U_2 = \mathrm{Gen}(A)$.

Next we state several propositions:

(25) Let S be a non void non empty many sorted signature, and let U_0 be a non-empty algebra over S, and let U_1 be a subalgebra of U_0, and let A, B be subsets of U_0. If $B = A \cup$ the sorts of U_1, then $\mathrm{Gen}(A) \bigsqcup U_1 = \mathrm{Gen}(B)$.

(26) Let S be a non void non empty many sorted signature, and let U_0 be a non-empty algebra over S, and let U_1 be a subalgebra of U_0, and let B be a subset of U_0. If B = the sorts of U_0, then $\mathrm{Gen}(B) \bigsqcup U_1 = \mathrm{Gen}(B)$.

(27) Let S be a non void non empty many sorted signature, and let U_0 be a non-empty algebra over S, and let U_1, U_2 be subalgebras of U_0. Then $U_1 \bigsqcup U_2 = U_2 \bigsqcup U_1$.

(28) Let S be a non void non empty many sorted signature, and let U_0 be a non-empty algebra over S, and let U_1, U_2 be strict subalgebras of U_0. Then $U_1 \cap (U_1 \bigsqcup U_2) = U_1$.

(29) Let S be a non void non empty many sorted signature with constant operations, and let U_0 be a non-empty algebra over S, and let U_1, U_2 be strict subalgebras of U_0. Then $U_1 \cap U_2 \bigsqcup U_2 = U_2$.

6. Lattice of Subalgebras of Many Sorted Algebra

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. The functor Subalgebras(U_0) yielding a non empty set is defined as follows:

(Def.20) For every x holds $x \in$ Subalgebras(U_0) iff x is a strict subalgebra of U_0.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. The functor MSAlgJoin(U_0) yields a binary operation on Subalgebras(U_0) and is defined by:

(Def.21) For all elements x, y of Subalgebras(U_0) and for all strict subalgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds (MSAlgJoin(U_0))(x, y) $= U_1 \bigsqcup U_2$.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. The functor MSAlgMeet(U_0) yielding a binary operation on Subalgebras(U_0) is defined by:

(Def.22) For all elements x, y of Subalgebras(U_0) and for all strict subalgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds (MSAlgMeet(U_0))(x, y) $= U_1 \cap U_2$.

In the sequel U_0 is a non-empty algebra over S.

We now state four propositions:

(30) MSAlgJoin(U_0) is commutative.

(31) MSAlgJoin(U_0) is associative.

(32) Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. Then MSAlgMeet(U_0) is commutative.

(33) Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. Then MSAlgMeet(U_0) is associative.

Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. The lattice of subalgebras of U_0 yields a strict lattice and is defined as follows:

(Def.23) The lattice of subalgebras of $U_0 = \langle$Subalgebras(U_0), MSAlgJoin(U_0), MSAlgMeet(U_0)$\rangle$.

The following proposition is true

(34) Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. Then the lattice of subalgebras of U_0 is bounded.

Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. Note that the lattice of subalgebras of U_0 is bounded.

We now state three propositions:

(35) Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. Then $\bot_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}(\mathrm{Constants}(U_0))$.

(36) Let S be a non void non empty many sorted signature with constant operations, and let U_0 be a non-empty algebra over S, and let B be a subset of U_0. If $B =$ the sorts of U_0, then $\top_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}(B)$.

(37) Let S be a non void non empty many sorted signature with constant operations and let U_0 be a strict non-empty algebra over S. Then $\top_{\text{the lattice of subalgebras of } U_0} = U_0$.

References

[1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[8] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[9] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[10] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.

[11] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[13] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[14] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

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Products of Many Sorted Algebras

Beata Madras
Warsaw University
Białystok

Summary. Product of two many sorted universal algebras and product of family of many sorted universal algebras are defined in this article. Operations on functions, such that commute, Frege, are also introduced.

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The papers [17], [18], [9], [10], [6], [7], [13], [11], [14], [4], [8], [2], [1], [3], [5], [16], [12], and [15] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity we follow the rules: I, J denote sets, A, B denote many sorted sets of I, i, j, x are arbitrary, and S denotes a non empty many sorted signature.

A set has common domain if:

(Def.1) For all functions f, g such that $f \in$ it and $g \in$ it holds $\operatorname{dom} f = \operatorname{dom} g$.

Let us mention that there exists a set which is functional and non empty and has common domain.

The following proposition is true

(1) $\{\emptyset\}$ is a functional set with common domain.

Let X be a functional set with common domain. The functor $\mathrm{DOM}(X)$ yielding a set is defined as follows:

(Def.2) (i) For every function x such that $x \in X$ holds $\mathrm{DOM}(X) = \operatorname{dom} x$ if $X \neq \emptyset$,

(ii) $\mathrm{DOM}(X) = \emptyset$, otherwise.

We now state the proposition

(2) For every functional set X with common domain such that $X = \{\emptyset\}$ holds $\mathrm{DOM}(X) = \emptyset$.

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Let I be a set and let M be a non-empty many sorted set of I. Observe that $\prod M$ is functional and non empty and has common domain.

2. Operations on Functions

The scheme *LambdaDMS* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

There exists a many sorted set X of $\mathcal{A}$ such that for every element d of $\mathcal{A}$ holds $X(d) = \mathcal{F}(d)$

for all values of the parameters.

Let f be a function. The functor commute(f) yields a function yielding function and is defined as follows:

(Def.5)[1] $\text{commute}(f) = \text{curry}' \, \text{uncurry} \, f$.

We now state several propositions:

(3) For every function f and for arbitrary x such that $x \in \text{dom}\,\text{commute}(f)$ holds $(\text{commute}(f))(x)$ is a function.

(4) For all sets A, B, C and for every function f such that $A \neq \emptyset$ and $B \neq \emptyset$ and $f \in (C^B)^A$ holds $\text{commute}(f) \in (C^A)^B$.

(5) Let A, B, C be sets and let f be a function. Suppose $A \neq \emptyset$ and $B \neq \emptyset$ and $f \in (C^B)^A$. Let g, h be functions and let x, y be arbitrary. Suppose $x \in A$ and $y \in B$ and $f(x) = g$ and $(\text{commute}(f))(y) = h$. Then $h(x) = g(y)$ and $\text{dom}\, h = A$ and $\text{dom}\, g = B$ and $\text{rng}\, h \subseteq C$ and $\text{rng}\, g \subseteq C$.

(6) For all sets A, B, C and for every function f such that $A \neq \emptyset$ and $B \neq \emptyset$ and $f \in (C^B)^A$ holds $\text{commute}(\text{commute}(f)) = f$.

(7) $\text{commute}(\square) = \square$.

Let F be a function. The functor $\blacksquare\text{commute}(F)$ yielding a function is defined by the conditions (Def.6).

(Def.6) (i) For every x holds $x \in \text{dom}\,\blacksquare\text{commute}(F)$ iff there exists a function f such that $f \in \text{dom}\, F$ and $x = \text{commute}(f)$, and

(ii) for every function f such that $f \in \text{dom}\,\blacksquare\text{commute}(F)$ holds $(\blacksquare\text{commute}(F))(f) = F(\text{commute}(f))$.

The following proposition is true

(8) For every function F such that $\text{dom}\, F = \{\emptyset\}$ holds $\blacksquare\text{commute}(F) = F$.

Let F be a function yielding function and let f be a function. The functor $F \looparrowleft f$ yielding a function is defined by:

(Def.7) $\text{dom}(F \looparrowleft f) = \text{dom}\, F$ and for arbitrary x and for every function g such that $x \in \text{dom}\, F$ and $g = F(x)$ holds $(F \looparrowleft f)(x) = g(f(x))$.

Let f be a function yielding function. The functor Frege(f) yields a many sorted function of $\prod(\text{dom}_\kappa\, f(\kappa))$ and is defined as follows:

[1] The definitions (Def.3) and (Def.4) have been removed.

(Def.8) For every function g such that $g \in \prod(\mathrm{dom}_\kappa f(\kappa))$ holds $(\mathrm{Frege}(f))(g) = f \leftarrow\!\!\!\!\!\!\circ\; g$.

Let us consider I, A, B. The functor $[\![A, B]\!]$ yielding a many sorted set of I is defined by:

(Def.9) For every i such that $i \in I$ holds $[\![A, B]\!](i) = [\!:A(i), B(i):\!]$.

Let us consider I and let A, B be non-empty many sorted sets of I. Note that $[\![A, B]\!]$ is non-empty.

Next we state the proposition

(9) Let I be a non empty set, and let J be a set, and let A, B be many sorted sets of I, and let f be a function from J into I. Then $[\![A, B]\!] \cdot f = [\![A \cdot f, B \cdot f]\!]$.

Let I be a non empty set, let us consider J, let A, B be non-empty many sorted sets of I, let p be a function from J into I^*, let r be a function from J into I, let j be arbitrary, let f be a function from $(A^\# \cdot p)(j)$ into $(A \cdot r)(j)$, and let g be a function from $(B^\# \cdot p)(j)$ into $(B \cdot r)(j)$. Let us assume that $j \in J$. The functor $]\!]f, g[\![$ yields a function from $([\![A, B]\!]^\# \cdot p)(j)$ into $([\![A, B]\!] \cdot r)(j)$ and is defined as follows:

(Def.10) For every function h such that $h \in ([\![A, B]\!]^\# \cdot p)(j)$ holds $]\!]f, g[\![(h) = \langle f(\mathrm{pr1}(h)), g(\mathrm{pr2}(h))\rangle$.

Let I be a non empty set, let us consider J, let A, B be non-empty many sorted sets of I, let p be a function from J into I^*, let r be a function from J into I, let F be a many sorted function from $A^\# \cdot p$ into $A \cdot r$, and let G be a many sorted function from $B^\# \cdot p$ into $B \cdot r$. The functor $]\!]F, G[\![$ yielding a many sorted function from $[\![A, B]\!]^\# \cdot p$ into $[\![A, B]\!] \cdot r$ is defined by the condition (Def.11).

(Def.11) Given j. Suppose $j \in J$. Let f be a function from $(A^\# \cdot p)(j)$ into $(A \cdot r)(j)$ and let g be a function from $(B^\# \cdot p)(j)$ into $(B \cdot r)(j)$. If $f = F(j)$ and $g = G(j)$, then $]\!]F, G[\![(j) =]\!]f, g[\![$.

3. Family of Many Sorted Universal Algebras

Let us consider I, S. A many sorted set of I is said to be an algebra family of I over S if:

(Def.12) For every i such that $i \in I$ holds $\mathrm{it}(i)$ is a non-empty algebra over S.

Let I be a non empty set, let us consider S, let A be an algebra family of I over S, and let i be an element of I. Then $A(i)$ is a non-empty algebra over S.

Let S be a non empty many sorted signature and let U_1 be a non-empty algebra over S. The functor $|U_1|$ yields a non empty set and is defined as follows:

(Def.13) $|U_1| = \bigcup \mathrm{rng}\,(\text{the sorts of } U_1)$.

Let I be a non empty set, let S be a non empty many sorted signature, and let A be an algebra family of I over S. The functor $|A|$ yields a non empty set and is defined as follows:

(Def.14) $|A| = \bigcup\{|A(i)| : i$ ranges over elements of $I\}$.

4. Product of Many Sorted Universal Algebras

We now state two propositions:

(10) Let S be a non void non empty many sorted signature, and let U_0 be an algebra over S, and let o be an operation symbol of S. Then $\mathrm{Args}(o, U_0) = \prod((\text{the sorts of } U_0) \cdot \mathrm{Arity}(o))$ and $\mathrm{dom}((\text{the sorts of } U_0) \cdot \mathrm{Arity}(o)) = \mathrm{dom}\,\mathrm{Arity}(o)$ and $\mathrm{Result}(o, U_0) = (\text{the sorts of } U_0)(\text{the result sort of } o)$.

(11) Let S be a non void non empty many sorted signature, and let U_0 be an algebra over S, and let o be an operation symbol of S. If $\mathrm{Arity}(o) = \varepsilon$, then $\mathrm{Args}(o, U_0) = \{\Box\}$.

Let us consider S and let U_1, U_2 be non-empty algebras over S. The functor $[\!:U_1, U_2:\!]$ yields a strict algebra over S and is defined as follows:

(Def.15) $[\!:U_1, U_2:\!] = \langle [\![\text{the sorts of } U_1, \text{the sorts of } U_2]\!], \rceil\!\rceil\text{the characteristics of } U_1, (\text{the characteristics of } U_2)\lceil\!\lceil \rangle$.

Let I be a non empty set, let us consider S, let s be a sort symbol of S, and let A be an algebra family of I over S. The functor $\mathrm{Carrier}(A, s)$ yielding a non-empty many sorted set of I is defined as follows:

(Def.16) For every element i of I holds $(\mathrm{Carrier}(A, s))(i) = (\text{the sorts of } A(i))(s)$.

Let I be a non empty set, let us consider S, and let A be an algebra family of I over S. The functor $\mathrm{SORTS}(A)$ yields a non-empty many sorted set of the carrier of S and is defined as follows:

(Def.17) For every sort symbol s of S holds $(\mathrm{SORTS}(A))(s) = \prod \mathrm{Carrier}(A, s)$.

Let I be a non empty set, let S be a non empty many sorted signature, and let A be an algebra family of I over S. The functor $\mathrm{OPER}(A)$ yields a many sorted function of I and is defined by:

(Def.18) For every element i of I holds $(\mathrm{OPER}(A))(i) =$ the characteristics of $A(i)$.

We now state two propositions:

(12) Let I be a non empty set, and let S be a non empty many sorted signature, and let A be an algebra family of I over S. Then $\mathrm{dom}\,\mathrm{uncurry}\,\mathrm{OPER}(A) = [\!:I, \text{the operation symbols of } S:\!]$.

(13) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S, and let o be an operation symbol of S. Then $\mathrm{commute}(\mathrm{OPER}(A)) \in ((\mathrm{rng}\,\mathrm{uncurry}\,\mathrm{OPER}(A))^I)^{\text{the operation symbols of } S}$.

Let I be a non empty set, let S be a non void non empty many sorted signature, let A be an algebra family of I over S, and let o be an operation symbol of S. The functor $A(o)$ yielding a many sorted function of I is defined by:

(Def.19) $A(o) = (\text{commute}(\text{OPER}(A)))(o)$.

We now state several propositions:

(14) Let I be a non empty set, and let i be an element of I, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S, and let o be an operation symbol of S. Then $A(o)(i) = \text{Den}(o, A(i))$.

(15) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S, and let o be an operation symbol of S, and let x be arbitrary. If $x \in \text{rng}\,\text{Frege}(A(o))$, then x is a function.

(16) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S, and let o be an operation symbol of S, and let f be a function. If $f \in \text{rng}\,\text{Frege}(A(o))$, then $\text{dom}\, f = I$ and for every element i of I holds $f(i) \in \text{Result}(o, A(i))$.

(17) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S, and let o be an operation symbol of S, and let f be a function. Suppose $f \in \text{dom}\,\text{Frege}(A(o))$. Then $\text{dom}\, f = I$ and for every element i of I holds $f(i) \in \text{Args}(o, A(i))$ and $\text{rng}\, f \subseteq |A|^{\text{dom}\,\text{Arity}(o)}$.

(18) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S, and let o be an operation symbol of S. Then $\text{dom}(\text{dom}_\kappa\, A(o)(\kappa)) = I$ and for every element i of I holds $(\text{dom}_\kappa\, A(o)(\kappa))(i) = \text{Args}(o, A(i))$.

Let I be a non empty set, let S be a non void non empty many sorted signature, and let A be an algebra family of I over S. The functor $\text{OPS}(A)$ yielding a many sorted function from $(\text{SORTS}(A))^{\#} \cdot (\text{the arity of } S)$ into $\text{SORTS}(A) \cdot (\text{the result sort of } S)$ is defined by:

(Def.20) For every operation symbol o of S holds $(\text{OPS}(A))(o) = (\text{Arity}(o) = \varepsilon \rightarrow \text{commute}(A(o)), \blacksquare\text{commute}(\text{Frege}(A(o))))$.

Let I be a non empty set, let S be a non void non empty many sorted signature, and let A be an algebra family of I over S. The functor $\prod A$ yields a strict algebra over S and is defined as follows:

(Def.21) $\prod A = \langle \text{SORTS}(A), \text{OPS}(A) \rangle$.

We now state two propositions:

(19) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S. Then $\prod A = \langle \text{SORTS}(A), \text{OPS}(A) \rangle$.

(20) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S. Then the

sorts of $\prod A = \mathrm{SORTS}(A)$ and the characteristics of $\prod A = \mathrm{OPS}(A)$.

References

[1] Grzegorz Bancerek. Cartesian product of functions. *Formalized Mathematics*, 2(**4**):547–552, 1991.
[2] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.
[3] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Formalized Mathematics*, 4(**1**):91–101, 1993.
[6] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[7] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.
[8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[11] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[12] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.
[13] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[14] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[15] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.
[16] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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Homomorphisms of Many Sorted Algebras

Małgorzata Korolkiewicz
Warsaw University
Białystok

Summary. The aim of this article is to present the definition and some properties of homomorphisms of many sorted algebras. Some auxiliary properties of many sorted functions also have been shown.

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The notation and terminology used in this paper have been introduced in the following articles: [10], [12], [13], [5], [6], [2], [4], [1], [11], [9], [7], [8], and [3].

1. Preliminaries

For simplicity we follow the rules: S is a non void non empty many sorted signature, U_1, U_2, U_3 are non-empty algebras over S, o is an operation symbol of S, and n is a natural number.

Let I be a non empty set, let A, B be non-empty many sorted sets of I, let F be a many sorted function from A into B, and let i be an element of I. Then $F(i)$ is a function from $A(i)$ into $B(i)$.

Let us consider S, U_1, U_2. A many sorted function from U_1 into U_2 is a many sorted function from the sorts of U_1 into the sorts of U_2.

Let I be a set and let A be a many sorted set of I. The functor id_A yields a many sorted function from A into A and is defined as follows:

(Def.1) For arbitrary i such that $i \in I$ holds $\mathrm{id}_A(i) = \mathrm{id}_{A(i)}$.

A function is "1-1" if:

(Def.2) For arbitrary i and for every function f such that $i \in \mathrm{dom}\,\mathrm{it}$ and $\mathrm{it}(i) = f$ holds f is one-to-one.

Let I be a set. Observe that there exists a many sorted function of I which is "1-1".

We now state the proposition

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(1) Let I be a set and let F be a many sorted function of I. Then F is "1-1" if and only if for arbitrary i and for every function f such that $i \in I$ and $F(i) = f$ holds f is one-to-one.

Let I be a set and let A, B be many sorted sets of I. A many sorted function from A into B is "onto" if:

(Def.3) For arbitrary i and for every function f from $A(i)$ into $B(i)$ such that $i \in I$ and $\mathrm{it}(i) = f$ holds $\mathrm{rng}\, f = B(i)$.

Let F, G be function yielding functions. The functor $G \circ F$ yielding a function yielding function is defined by the conditions (Def.4).

(Def.4) (i) $\mathrm{dom}(G \circ F) = \mathrm{dom}\, F \cap \mathrm{dom}\, G$, and

(ii) for arbitrary i and for every function f and for every function g such that $i \in \mathrm{dom}(G \circ F)$ and $f = F(i)$ and $g = G(i)$ holds $(G \circ F)(i) = g \cdot f$.

We now state the proposition

(2) Let I be a set, and let A be a many sorted set of I, and let B, C be non-empty many sorted sets of I, and let F be a many sorted function from A into B, and let G be a many sorted function from B into C. Then

(i) $\mathrm{dom}(G \circ F) = I$, and

(ii) for arbitrary i and for every function f from $A(i)$ into $B(i)$ and for every function g from $B(i)$ into $C(i)$ such that $i \in I$ and $f = F(i)$ and $g = G(i)$ holds $(G \circ F)(i) = g \cdot f$.

Let I be a set, let A be a many sorted set of I, let B, C be non-empty many sorted sets of I, let F be a many sorted function from A into B, and let G be a many sorted function from B into C. Then $G \circ F$ is a many sorted function from A into C.

Next we state two propositions:

(3) Let I be a set, and let A, B be non-empty many sorted sets of I, and let F be a many sorted function from A into B. Then $F \circ \mathrm{id}_A = F$.

(4) Let I be a set, and let A be a many sorted set of I, and let B be a non-empty many sorted set of I, and let F be a many sorted function from A into B. Then $\mathrm{id}_B \circ F = F$.

Let I be a set, let A, B be non-empty many sorted sets of I, and let F be a many sorted function from A into B. Let us assume that F is "1-1" and "onto". The functor F^{-1} yielding a many sorted function from B into A is defined as follows:

(Def.5) For arbitrary i and for every function f from $A(i)$ into $B(i)$ such that $i \in I$ and $f = F(i)$ holds $F^{-1}(i) = f^{-1}$.

We now state the proposition

(5) Let I be a set, and let A, B be non-empty many sorted sets of I, and let H be a many sorted function from A into B, and let H_1 be a many sorted function from B into A. If H is "1-1" and "onto" and $H_1 = H^{-1}$, then $H \circ H_1 = \mathrm{id}_B$ and $H_1 \circ H = \mathrm{id}_A$.

Let I be a set, let A be a many sorted set of I, and let F be a many sorted function of I. The functor $F \circ A$ yields a many sorted set of I and is defined as follows:

(Def.6) For arbitrary i and for every function f such that $i \in I$ and $f = F(i)$ holds $(F \circ A)(i) = f^{\circ}A(i)$.

Let us consider S, U_1, o. Observe that every element of $\mathrm{Args}(o, U_1)$ is function-like and relation-like.

2. Homomorphisms of Many Sorted Algebras

One can prove the following proposition

(6) Let x be an element of $\mathrm{Args}(o, U_1)$. Then $\mathrm{dom}\, x = \mathrm{dom}\,\mathrm{Arity}(o)$ and for arbitrary y such that $y \in \mathrm{dom}((\text{the sorts of } U_1) \cdot \mathrm{Arity}(o))$ holds $x(y) \in ((\text{the sorts of } U_1) \cdot \mathrm{Arity}(o))(y)$.

Let us consider S, U_1, U_2, o, let F be a many sorted function from U_1 into U_2, and let x be an element of $\mathrm{Args}(o, U_1)$. The functor $F\#x$ yielding an element of $\mathrm{Args}(o, U_2)$ is defined by:

(Def.7) For every n such that $n \in \mathrm{dom}\, x$ holds $(F\#x)(n) = F(\pi_n \mathrm{Arity}(o))(x(n))$.

The following two propositions are true:

(7) For all S, o, U_1 and for every element x of $\mathrm{Args}(o, U_1)$ holds $x = \mathrm{id}_{(\text{the sorts of } U_1)}\#x$.

(8) Let H_1 be a many sorted function from U_1 into U_2, and let H_2 be a many sorted function from U_2 into U_3, and let x be an element of $\mathrm{Args}(o, U_1)$. Then $(H_2 \circ H_1)\#x = H_2\#(H_1\#x)$.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. We say that F is a homomorphism of U_1 into U_2 if and only if:

(Def.8) For every operation symbol o of S and for every element x of $\mathrm{Args}(o, U_1)$ holds $F(\text{the result sort of } o)((\mathrm{Den}(o, U_1))(x)) = (\mathrm{Den}(o, U_2))(F\#x)$.

Next we state two propositions:

(9) $\mathrm{id}_{(\text{the sorts of } U_1)}$ is a homomorphism of U_1 into U_1.

(10) Let H_1 be a many sorted function from U_1 into U_2 and let H_2 be a many sorted function from U_2 into U_3. Suppose H_1 is a homomorphism of U_1 into U_2 and H_2 is a homomorphism of U_2 into U_3. Then $H_2 \circ H_1$ is a homomorphism of U_1 into U_3.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. We say that F is an epimorphism of U_1 onto U_2 if and only if:

(Def.9) F is a homomorphism of U_1 into U_2 and "onto".

One can prove the following proposition

(11) Let F be a many sorted function from U_1 into U_2 and let G be a many sorted function from U_2 into U_3. Suppose F is an epimorphism of U_1 onto U_2 and G is an epimorphism of U_2 onto U_3. Then $G \circ F$ is an epimorphism of U_1 onto U_3.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. We say that F is a monomorphism of U_1 into U_2 if and only if:

(Def.10) F is a homomorphism of U_1 into U_2 and "1-1".

The following proposition is true

(12) Let F be a many sorted function from U_1 into U_2 and let G be a many sorted function from U_2 into U_3. Suppose F is a monomorphism of U_1 into U_2 and G is a monomorphism of U_2 into U_3. Then $G \circ F$ is a monomorphism of U_1 into U_3.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. We say that F is an isomorphism of U_1 and U_2 if and only if:

(Def.11) F is an epimorphism of U_1 onto U_2 and a monomorphism of U_1 into U_2.

The following propositions are true:

(13) Let F be a many sorted function from U_1 into U_2. Then F is an isomorphism of U_1 and U_2 if and only if F is a homomorphism of U_1 into U_2 "onto" and "1-1".

(14) Let H be a many sorted function from U_1 into U_2 and let H_1 be a many sorted function from U_2 into U_1. Suppose H is an isomorphism of U_1 and U_2 and $H_1 = H^{-1}$. Then H_1 is an isomorphism of U_2 and U_1.

(15) Let H be a many sorted function from U_1 into U_2 and let H_1 be a many sorted function from U_2 into U_3. Suppose H is an isomorphism of U_1 and U_2 and H_1 is an isomorphism of U_2 and U_3. Then $H_1 \circ H$ is an isomorphism of U_1 and U_3.

Let us consider S, U_1, U_2. We say that U_1 and U_2 are isomorphic if and only if:

(Def.12) There exists many sorted function from U_1 into U_2 which is an isomorphism of U_1 and U_2.

Next we state three propositions:

(16) U_1 and U_1 are isomorphic.

(17) If U_1 and U_2 are isomorphic, then U_2 and U_1 are isomorphic.

(18) If U_1 and U_2 are isomorphic and U_2 and U_3 are isomorphic, then U_1 and U_3 are isomorphic.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. Let us assume that F is a homomorphism of U_1 into U_2. The functor $\mathrm{Im}\, F$ yields a strict non-empty subalgebra of U_2 and is defined as follows:

(Def.13) The sorts of $\mathrm{Im}\, F = F^{\circ}$ (the sorts of U_1).

We now state several propositions:

(19) Let U_2 be a strict non-empty algebra over S and let F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then F is an epimorphism of U_1 onto U_2 if and only if $\operatorname{Im} F = U_2$.

(20) Let F be a many sorted function from U_1 into U_2 and let G be a many sorted function from U_1 into $\operatorname{Im} F$. Suppose $F = G$ and F is a homomorphism of U_1 into U_2. Then G is an epimorphism of U_1 onto $\operatorname{Im} F$.

(21) Let F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then there exists a many sorted function G from U_1 into $\operatorname{Im} F$ such that $F = G$ and G is an epimorphism of U_1 onto $\operatorname{Im} F$.

(22) Let U_2 be a strict non-empty subalgebra of U_1 and let G be a many sorted function from U_2 into U_1. If $G = \mathrm{id}_{(\text{the sorts of } U_2)}$, then G is a monomorphism of U_2 into U_1.

(23) Let F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then there exists a many sorted function F_1 from U_1 into $\operatorname{Im} F$ and there exists a many sorted function F_2 from $\operatorname{Im} F$ into U_2 such that F_1 is an epimorphism of U_1 onto $\operatorname{Im} F$ and F_2 is a monomorphism of $\operatorname{Im} F$ into U_2 and $F = F_2 \circ F_1$.

References

[1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Formalized Mathematics*, 5(**1**):47–54, 1996.
[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.
[8] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.
[9] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[11] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[12] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[13] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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Free Many Sorted Universal Algebra

Beata Perkowska
Warsaw University
Białystok

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The terminology and notation used in this paper are introduced in the following papers: [21], [24], [25], [11], [22], [12], [7], [18], [13], [10], [2], [4], [5], [23], [14], [6], [1], [16], [3], [8], [20], [17], [19], [9], and [15].

1. Preliminaries

The following proposition is true

(1) Let I be a set, and let J be a non empty set, and let f be a function from I into J^*, and let X be a many sorted set of J, and let p be an element of J^*, and let x be arbitrary. If $x \in I$ and $p = f(x)$, then $(X^\# \cdot f)(x) = \prod(X \cdot p)$.

Let I be a set, let A, B be many sorted sets of I, let C be a many sorted subset of A, and let F be a many sorted function from A into B. The functor $F \restriction C$ yielding a many sorted function from C into B is defined as follows:

(Def.1) For arbitrary i such that $i \in I$ and for every function f from $A(i)$ into $B(i)$ such that $f = F(i)$ holds $(F \restriction C)(i) = f \restriction C(i)$.

Let I be a set, let X be a many sorted set of I, and let i be arbitrary. Let us assume that $i \in I$. The functor $\mathrm{coprod}(i, X)$ yields a set and is defined as follows:

(Def.2) For arbitrary x holds $x \in \mathrm{coprod}(i, X)$ iff there exists arbitrary a such that $a \in X(i)$ and $x = \langle a, i \rangle$.

Let I be a set and let X be a many sorted set of I. Then $\mathrm{disjoint}\, X$ is a many sorted set of I and it can be characterized by the condition:

(Def.3) For arbitrary i such that $i \in I$ holds $(\mathrm{disjoint}\, X)(i) = \mathrm{coprod}(i, X)$.

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We introduce coprod(X) as a synonym of disjoint X.

Let I be a non empty set and let X be a non-empty many sorted set of I. One can verify that coprod(X) is non-empty.

Let I be a non empty set and let X be a non-empty many sorted set of I. One can check that $\bigcup X$ is non empty.

We now state the proposition

(2) Let I be a set, and let X be a many sorted set of I, and let i be arbitrary. If $i \in I$, then $X(i) \neq \emptyset$ iff $(\mathrm{coprod}(X))(i) \neq \emptyset$.

2. Free Many Sorted Universal Algebra - General Notions

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. A subset of U_0 is said to be a generator set of U_0 if:

(Def.4) The sorts of Gen(it) = the sorts of U_0.

Next we state the proposition

(3) Let S be a non void non empty many sorted signature, and let U_0 be a strict non-empty algebra over S, and let A be a subset of U_0. Then A is a generator set of U_0 if and only if $\mathrm{Gen}(A) = U_0$.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. A generator set of U_0 is free if it satisfies the condition (Def.5).

(Def.5) Let U_1 be a non-empty algebra over S and let f be a many sorted function from it into the sorts of U_1. Then there exists a many sorted function h from U_0 into U_1 such that h is a homomorphism of U_0 into U_1 and $h \restriction \mathrm{it} = f$.

Let S be a non void non empty many sorted signature. A non-empty algebra over S is free if:

(Def.6) There exists generator set of it which is free.

The following proposition is true

(4) Let S be a non void non empty many sorted signature and let X be a many sorted set of the carrier of S. Then $\bigcup \mathrm{coprod}(X) \cap [\!:$ the operation symbols of S, $\{$the carrier of $S\}:\!] = \emptyset$.

3. Semidisjoint Many Sorted Signature

Let S be a non void many sorted signature. Note that the operation symbols of S is non empty.

Let S be a non void non empty many sorted signature and let X be a many sorted set of the carrier of S. The functor REL(X) yields a relation between $[\!:$ the operation symbols of S, $\{$the carrier of $S\}:\!] \cup \bigcup \mathrm{coprod}(X)$ and

$([\!:$ the operation symbols of S, $\{$the carrier of $S\}$ $:\!] \cup \bigcup \mathrm{coprod}(X))^*$ and is defined by the condition (Def.9).

(Def.9)[1] Let a be an element of $[\!:$ the operation symbols of S, $\{$the carrier of $S\}$ $:\!] \cup \bigcup \mathrm{coprod}(X)$ and let b be an element of $([\!:$ the operation symbols of S, $\{$the carrier of $S\}$ $:\!] \cup \bigcup \mathrm{coprod}(X))^*$. Then $\langle a, b \rangle \in \mathrm{REL}(X)$ if and only if the following conditions are satisfied:

(i) $a \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}$ $:\!]$, and

(ii) for every operation symbol o of S such that $\langle o$, the carrier of $S\rangle = a$ holds $\mathrm{len}\, b = \mathrm{len}\, \mathrm{Arity}(o)$ and for arbitrary x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}$ $:\!]$, then for every operation symbol o_1 of S such that $\langle o_1$, the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 = \mathrm{Arity}(o)(x)$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then $b(x) \in \mathrm{coprod}(\mathrm{Arity}(o)(x), X)$.

In the sequel S will be a non void non empty many sorted signature, X will be a many sorted set of the carrier of S, o will be an operation symbol of S, and b will be an element of $([\!:$ the operation symbols of S, $\{$the carrier of $S\}$ $:\!] \cup \bigcup \mathrm{coprod}(X))^*$.

Next we state the proposition

(5) $\langle\langle o$, the carrier of $S\rangle, b\rangle \in \mathrm{REL}(X)$ if and only if the following conditions are satisfied:

(i) $\mathrm{len}\, b = \mathrm{len}\, \mathrm{Arity}(o)$, and

(ii) for arbitrary x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}$ $:\!]$, then for every operation symbol o_1 of S such that $\langle o_1$, the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 = \mathrm{Arity}(o)(x)$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then $b(x) \in \mathrm{coprod}(\mathrm{Arity}(o)(x), X)$.

Let S be a non void non empty many sorted signature and let X be a many sorted set of the carrier of S. The functor $\mathrm{DTConMSA}(X)$ yielding a strict tree construction structure is defined as follows:

(Def.10) $\mathrm{DTConMSA}(X) = \langle [\!:$ the operation symbols of S, $\{$the carrier of $S\}$ $:\!] \cup \bigcup \mathrm{coprod}(X), \mathrm{REL}(X)\rangle$.

Let S be a non void non empty many sorted signature and let X be a many sorted set of the carrier of S. Observe that $\mathrm{DTConMSA}(X)$ is non empty.

We now state the proposition

(6) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Then the nonterminals of $\mathrm{DTConMSA}(X) = [\!:$ the operation symbols of S, $\{$the carrier of $S\}$ $:\!]$ and the terminals of $\mathrm{DTConMSA}(X) = \bigcup \mathrm{coprod}(X)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Observe that $\mathrm{DTConMSA}(X)$ has terminals, nonterminals, and useful nonterminals.

One can prove the following proposition

[1]The definitions (Def.7) and (Def.8) have been removed.

(7) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let t be arbitrary. Then $t \in$ the terminals of DTConMSA(X) if and only if there exists a sort symbol s of S and there exists arbitrary x such that $x \in X(s)$ and $t = \langle x, s \rangle$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, and let o be an operation symbol of S. The functor Sym(o, X) yielding a symbol of DTConMSA(X) is defined by:

(Def.11) $\mathrm{Sym}(o, X) = \langle o$, the carrier of $S \rangle$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, and let s be a sort symbol of S. The functor FreeSort(X, s) yielding a non empty subset of TS(DTConMSA(X)) is defined by the condition (Def.12).

(Def.12) $\mathrm{FreeSort}(X, s) = \{a : a$ ranges over elements of TS(DTConMSA(X)), $\bigvee_x \; x \in X(s) \;\wedge\; a =$ the root tree of $\langle x, s \rangle \;\vee\; \bigvee_o \; \langle o$, the carrier of $S \rangle = a(\varepsilon) \;\wedge\;$ the result sort of $o = s\}$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. The functor FreeSorts(X) yielding a non-empty many sorted set of the carrier of S is defined by:

(Def.13) For every sort symbol s of S holds $(\mathrm{FreeSorts}(X))(s) = \mathrm{FreeSort}(X, s)$.

The following propositions are true:

(8) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let o be an operation symbol of S, and let x be arbitrary. Suppose $x \in ((\mathrm{FreeSorts}(X))^{\#} \cdot (\text{the arity of } S))(o)$. Then x is a finite sequence of elements of TS(DTConMSA(X)).

(9) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let o be an operation symbol of S, and let p be a finite sequence of elements of TS(DTConMSA(X)). Then $p \in ((\mathrm{FreeSorts}(X))^{\#} \cdot (\text{the arity of } S))(o)$ if and only if $\mathrm{dom}\, p = \mathrm{dom}\, \mathrm{Arity}(o)$ and for every natural number n such that $n \in \mathrm{dom}\, p$ holds $p(n) \in \mathrm{FreeSort}(X, \pi_n \, \mathrm{Arity}(o))$.

(10) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let o be an operation symbol of S, and let p be a finite sequence of elements of TS(DTConMSA(X)). Then $\mathrm{Sym}(o, X) \Rightarrow$ the roots of p if and only if $p \in ((\mathrm{FreeSorts}(X))^{\#} \cdot (\text{the arity of } S))(o)$.

(11) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let o be an operation symbol of S. Then $(\mathrm{FreeSorts}(X) \cdot (\text{the result sort of } S))(o) \neq \emptyset$.

(12) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Then $\bigcup \mathrm{rng}\, \mathrm{FreeSorts}(X) =$ TS(DTConMSA(X)).

(13) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let s_1, s_2 be sort symbols of S. If $s_1 \neq s_2$, then $(\mathrm{FreeSorts}(X))(s_1) \cap (\mathrm{FreeSorts}(X))(s_2) = \emptyset$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, and let o be an operation symbol of S. The functor $\mathrm{DenOp}(o, X)$ yielding a function from $((\mathrm{FreeSorts}(X))^{\#} \cdot (\text{the arity of } S))(o)$ into $(\mathrm{FreeSorts}(X) \cdot (\text{the result sort of } S))(o)$ is defined by:

(Def.14) For every finite sequence p of elements of $\mathrm{TS}(\mathrm{DTConMSA}(X))$ such that $\mathrm{Sym}(o, X) \Rightarrow$ the roots of p holds $(\mathrm{DenOp}(o, X))(p) = \mathrm{Sym}(o, X)\text{-tree}(p)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. The functor $\mathrm{FreeOperations}(X)$ yielding a many sorted function from $(\mathrm{FreeSorts}(X))^{\#} \cdot (\text{the arity of } S)$ into $\mathrm{FreeSorts}(X) \cdot (\text{the result sort of } S)$ is defined as follows:

(Def.15) For every operation symbol o of S holds $(\mathrm{FreeOperations}(X))(o) = \mathrm{DenOp}(o, X)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. The functor $\mathrm{Free}(X)$ yields a strict non-empty algebra over S and is defined by:

(Def.16) $\mathrm{Free}(X) = \langle \mathrm{FreeSorts}(X), \mathrm{FreeOperations}(X) \rangle$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, and let s be a sort symbol of S. The functor $\mathrm{FreeGenerator}(s, X)$ yields a non empty subset of $(\mathrm{FreeSorts}(X))(s)$ and is defined as follows:

(Def.17) For arbitrary x holds $x \in \mathrm{FreeGenerator}(s, X)$ iff there exists arbitrary a such that $a \in X(s)$ and $x =$ the root tree of $\langle a, s \rangle$.

The following proposition is true

(14) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S, and let s be a sort symbol of S. Then $\mathrm{FreeGenerator}(s, X) = \{$the root tree of t: t ranges over symbols of $\mathrm{DTConMSA}(X)$, $t \in$ the terminals of $\mathrm{DTConMSA}(X) \ \wedge\ t_{\mathbf{2}} = s\}$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. The functor $\mathrm{FreeGenerator}(X)$ yielding a generator set of $\mathrm{Free}(X)$ is defined as follows:

(Def.18) For every sort symbol s of S holds $(\mathrm{FreeGenerator}(X))(s) = \mathrm{FreeGenerator}(s, X)$.

We now state two propositions:

(15) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Then $\mathrm{FreeGenerator}(X)$ is non-empty.

(16) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Then

$\bigcup$rng FreeGenerator(X) = {the root tree of t: t ranges over symbols of DTConMSA(X), $t \in$ the terminals of DTConMSA(X)}.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, and let s be a sort symbol of S. The functor Reverse(s, X) yielding a function from FreeGenerator(s, X) into $X(s)$ is defined as follows:

(Def.19) For every symbol t of DTConMSA(X) such that the root tree of $t \in$ FreeGenerator(s, X) holds (Reverse(s, X))(the root tree of t) = $t_{\mathbf{1}}$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. The functor Reverse(X) yielding a many sorted function from FreeGenerator(X) into X is defined by:

(Def.20) For every sort symbol s of S holds (Reverse(X))(s) = Reverse(s, X).

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, let A be a non-empty many sorted set of the carrier of S, let F be a many sorted function from FreeGenerator(X) into A, and let t be a symbol of DTConMSA(X). Let us assume that $t \in$ the terminals of DTConMSA(X). The functor $\pi(F, A, t)$ yielding an element of $\bigcup A$ is defined as follows:

(Def.21) For every function f such that $f = F(t_{\mathbf{2}})$ holds $\pi(F, A, t) = f$(the root tree of t).

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S, and let t be a symbol of DTConMSA(X). Let us assume that there exists a finite sequence p such that $t \Rightarrow p$. The functor ${}^{@}(X, t)$ yielding an operation symbol of S is defined by:

(Def.22) $\langle {}^{@}(X, t)$, the carrier of $S \rangle = t$.

Let S be a non void non empty many sorted signature, let U_0 be a non-empty algebra over S, let o be an operation symbol of S, and let p be a finite sequence. Let us assume that $p \in$ Args(o, U_0). The functor $\pi(o, U_0, p)$ yielding an element of $\bigcup$(the sorts of U_0) is defined by:

(Def.23) $\pi(o, U_0, p) = (\mathrm{Den}(o, U_0))(p)$.

Next we state two propositions:

(17) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Then FreeGenerator(X) is free.

(18) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S. Then Free(X) is free.

Let S be a non void non empty many sorted signature. One can check that there exists a non-empty algebra over S which is free and strict.

Let S be a non void non empty many sorted signature and let U_0 be a free non-empty algebra over S. One can verify that there exists a generator set of U_0 which is free.

One can prove the following propositions:

(19) Let S be a non void non empty many sorted signature and let U_1 be a non-empty algebra over S. Then there exists a strict free non-empty algebra U_0 over S such that there exists many sorted function from U_0 into U_1 which is an epimorphism of U_0 onto U_1.

(20) Let S be a non void non empty many sorted signature and let U_1 be a strict non-empty algebra over S. Then there exists a strict free non-empty algebra U_0 over S and there exists a many sorted function F from U_0 into U_1 such that F is an epimorphism of U_0 onto U_1 and $\mathrm{Im}\, F = U_1$.

References

[1] Grzegorz Bancerek. Cartesian product of functions. *Formalized Mathematics*, 2(**4**):547–552, 1991.

[2] Grzegorz Bancerek. Introduction to trees. *Formalized Mathematics*, 1(**2**):421–427, 1990.

[3] Grzegorz Bancerek. Joining of decorated trees. *Formalized Mathematics*, 4(**1**):77–82, 1993.

[4] Grzegorz Bancerek. König's lemma. *Formalized Mathematics*, 2(**3**):397–402, 1991.

[5] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[6] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Formalized Mathematics*, 3(**2**):195–204, 1992.

[7] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[8] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Formalized Mathematics*, 4(**1**):91–101, 1993.

[9] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Formalized Mathematics*, 5(**1**):47–54, 1996.

[10] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[11] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[12] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[13] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[14] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar - part 1. *Formalized Mathematics*, 2(**5**):683–687, 1991.

[15] Mał gorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Formalized Mathematics*, 5(**1**):61–65, 1996.

[16] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Formalized Mathematics*, 3(**2**):251–253, 1992.

[17] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.

[18] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[19] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.

[20] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.

[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[22] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[23] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[25] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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T_0 Topological Spaces

Mariusz Żynel
Warsaw University
Białystok

Adam Guzowski
Warsaw University
Białystok

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The papers [7], [10], [9], [1], [2], [4], [3], [6], [5], and [8] provide the terminology and notation for this paper.

The following two propositions are true:

(1) Let A, B be non empty sets and let R_1, R_2 be relations between A and B. Suppose that for every element x of A and for every element y of B holds $\langle x, y\rangle \in R_1$ iff $\langle x, y\rangle \in R_2$. Then $R_1 = R_2$.

(2) Let X, Y be non empty sets, and let f be a function from X into Y, and let A be a subset of X. Suppose that for all elements x_1, x_2 of X such that $x_1 \in A$ and $f(x_1) = f(x_2)$ holds $x_2 \in A$. Then $f^{-1} f^{\circ} A = A$.

Let T, S be topological spaces. We say that T and S are homeomorphic if and only if:

(Def.1) There exists map from T into S which is a homeomorphism.

Let T, S be topological spaces and let f be a map from T into S. We say that f is open if and only if:

(Def.2) For every subset A of T such that A is open holds $f^{\circ} A$ is open.

Let T be a topological space. The functor Indiscernibility(T) yielding an equivalence relation of the carrier of T is defined by the condition (Def.3).

(Def.3) Let p, q be points of T. Then $\langle p, q\rangle \in$ Indiscernibility(T) if and only if for every subset A of T such that A is open holds $p \in A$ iff $q \in A$.

Let T be a topological space. The functor $T_{/\,\mathrm{Indiscernibility}\,T}$ yields a non empty partition of the carrier of T and is defined as follows:

(Def.4) $T_{/\,\mathrm{Indiscernibility}\,T} = \mathrm{Classes}\,\mathrm{Indiscernibility}(T)$.

Let T be a topological space. The functor T_0-reflex(T) yields a topological space and is defined as follows:

(Def.5) T_0-reflex(T) = the decomposition space of $T_{/\,\mathrm{Indiscernibility}\,T}$.

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Let T be a topological space. The functor T_0-map(T) yielding a continuous map from T into T_0-reflex(T) is defined as follows:

(Def.6) T_0-map(T) = the projection onto $T_{/\,\text{Indiscernibility}\,T}$.

One can prove the following propositions:

(3) For every topological space T and for every point p of T holds $p \in (T_0\text{-map}(T))(p)$.

(4) For every topological space T holds dom T_0-map(T) = the carrier of T and rng T_0-map(T) $\subseteq$ the carrier of T_0-reflex(T).

(5) Let T be a topological space. Then the carrier of T_0-reflex(T) = $T_{/\,\text{Indiscernibility}\,T}$ and the topology of T_0-reflex(T) = $\{A : A$ ranges over subsets of $T_{/\,\text{Indiscernibility}\,T}$, $\bigcup A \in$ the topology of $T\}$.

(6) For every topological space T and for every subset V of T_0-reflex(T) holds V is open iff $\bigcup V \in$ the topology of T.

(7) Let T be a topological space and let C be arbitrary. Then C is a point of T_0-reflex(T) if and only if there exists a point p of T such that $C = [p]_{\text{Indiscernibility}(T)}$.

(8) For every topological space T and for every point p of T holds $(T_0\text{-map}(T))(p) = [p]_{\text{Indiscernibility}(T)}$.

(9) For every topological space T and for all points p, q of T holds $(T_0\text{-map}(T))(q) = (T_0\text{-map}(T))(p)$ iff $\langle q, p\rangle \in$ Indiscernibility(T).

(10) Let T be a topological space and let A be a subset of T. Suppose A is open. Let p, q be points of T. If $p \in A$ and $(T_0\text{-map}(T))(p) = (T_0\text{-map}(T))(q)$, then $q \in A$.

(11) Let T be a topological space and let A be a subset of T. Suppose A is open. Let C be a subset of T. If $C \in T_{/\,\text{Indiscernibility}\,T}$ and C meets A, then $C \subseteq A$.

(12) For every topological space T holds T_0-map(T) is open.

A topological structure is discernible if it satisfies the condition (Def.7).

(Def.7) Let x, y be points of it. Suppose $x \neq y$. Then there exists a subset V of it such that V is open but $x \in V$ and $y \notin V$ or $y \in V$ and $x \notin V$.

Let us note that there exists a topological space which is discernible.

A T_0-space is a discernible topological space.

One can prove the following propositions:

(13) For every topological space T holds T_0-reflex(T) is a T_0-space.

(14) Let T, S be topological spaces. Given a map h from T_0-reflex(S) into T_0-reflex(T) such that h is a homeomorphism and T_0-map(T) and $h \cdot T_0$-map(S) are fiberwise equipotent. Then T and S are homeomorphic.

(15) Let T be a topological space, and let T_0 be a T_0-space, and let f be a continuous map from T into T_0, and let p, q be points of T. If $\langle p, q\rangle \in$ Indiscernibility(T), then $f(p) = f(q)$.

(16) Let T be a topological space, and let T_0 be a T_0-space, and let f be a continuous map from T into T_0, and let p be a point of T. Then $f^\circ([p]_{\mathrm{Indiscernibility}(T)}) = \{f(p)\}$.

(17) Let T be a topological space, and let T_0 be a T_0-space, and let f be a continuous map from T into T_0. Then there exists a continuous map h from T_0-reflex(T) into T_0 such that $f = h \cdot T_0$-map(T).

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[4] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[5] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.

[6] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[8] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[9] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[10] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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Many Sorted Quotient Algebra

Małgorzata Korolkiewicz
Warsaw University
Białystok

Summary. This article introduces the construction of a many sorted quotient algebra. A few preliminary notions such as a many sorted relation, a many sorted equivalence relation, a many sorted congruence and the set of all classes of a many sorted relation are also formulated.

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The notation and terminology used here are introduced in the following papers: [13], [15], [5], [16], [10], [6], [2], [4], [1], [14], [12], [8], [11], [3], [7], and [9].

1. Many Sorted Relation

In this paper S will be a non void non empty many sorted signature and o will be an operation symbol of S.

A function is binary relation yielding if:

(Def.1) For arbitrary x such that $x \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(x)$ is a binary relation.

Let I be a set. Observe that there exists a many sorted set of I which is binary relation yielding.

Let I be a set. A many sorted relation of I is a binary relation yielding many sorted set of I.

Let I be a set and let A, B be many sorted sets of I. A many sorted set of I is said to be a many sorted relation between A and B if:

(Def.2) For arbitrary i such that $i \in I$ holds $\mathrm{it}(i)$ is a relation between $A(i)$ and $B(i)$.

Let I be a set and let A, B be many sorted sets of I. Note that every many sorted relation between A and B is binary relation yielding.

Let I be a set and let A be a many sorted set of I. A many sorted relation of A is a many sorted relation between A and A.

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Let I be a set and let A be a many sorted set of I. A many sorted relation of A is equivalence if it satisfies the condition (Def.3).

(Def.3) Let i be arbitrary and let R be a binary relation on $A(i)$. If $i \in I$ and $\mathrm{it}(i) = R$, then R is an equivalence relation of $A(i)$.

Let I be a non empty set, let A, B be many sorted sets of I, let F be a many sorted relation between A and B, and let i be an element of I. Then $F(i)$ is a relation between $A(i)$ and $B(i)$.

Let S be a non empty many sorted signature and let U_1 be an algebra over S.

(Def.4) A many sorted relation of the sorts of U_1 is said to be a many sorted relation of U_1.

Let S be a non empty many sorted signature and let U_1 be an algebra over S. A many sorted relation of U_1 is equivalence if:

(Def.5) It is equivalence.

Let S be a non void non empty many sorted signature and let U_1 be an algebra over S. Note that there exists a many sorted relation of U_1 which is equivalence.

One can prove the following proposition

(1) Let S be a non void non empty many sorted signature, and let U_1 be an algebra over S, and let R be an equivalence many sorted relation of U_1, and let s be a sort symbol of S. Then $R(s)$ is an equivalence relation of (the sorts of U_1)(s).

Let S be a non void non empty many sorted signature and let U_1 be a non-empty algebra over S. An equivalence many sorted relation of U_1 is called a congruence of U_1 if it satisfies the condition (Def.6).

(Def.6) Let o be an operation symbol of S and let x, y be elements of $\mathrm{Args}(o, U_1)$. Suppose that for every natural number n such that $n \in \mathrm{dom}\, x$ holds $\langle x(n), y(n) \rangle \in \mathrm{it}(\pi_n \mathrm{Arity}(o))$. Then $\langle (\mathrm{Den}(o, U_1))(x), (\mathrm{Den}(o, U_1))(y) \rangle \in \mathrm{it}$(the result sort of o).

Let S be a non void non empty many sorted signature, let U_1 be an algebra over S, let R be an equivalence many sorted relation of U_1, and let i be an element of the carrier of S. Then $R(i)$ is an equivalence relation of (the sorts of U_1)(i).

Let S be a non void non empty many sorted signature, let U_1 be an algebra over S, let R be an equivalence many sorted relation of U_1, let i be an element of the carrier of S, and let x be an element of (the sorts of U_1)(i). The functor $[x]_R$ yields a subset of (the sorts of U_1)(i) and is defined by:

(Def.7) $[x]_R = [x]_{R(i)}$.

Let us consider S, let U_1 be a non-empty algebra over S, and let R be a congruence of U_1. The functor $\mathrm{Classes}\, R$ yields a non-empty many sorted set of the carrier of S and is defined by:

(Def.8) For every element s of the carrier of S holds $(\mathrm{Classes}\, R)(s) = \mathrm{Classes}\, R(s)$.

2. Many Sorted Quotient Algebra

Let us consider S, let M_1, M_2 be many sorted sets of the operation symbols of S, let F be a many sorted function from M_1 into M_2, and let o be an operation symbol of S. Then $F(o)$ is a function from $M_1(o)$ into $M_2(o)$.

Let I be a non empty set, let p be a finite sequence of elements of I, and let X be a non-empty many sorted set of I. Then $X \cdot p$ is a non-empty many sorted set of $\mathrm{dom}\, p$.

Let us consider S, o, let A be a non-empty algebra over S, let R be a congruence of A, and let x be an element of $\mathrm{Args}(o, A)$. The functor $R\#x$ yields an element of $\prod(\mathrm{Classes}\, R \cdot \mathrm{Arity}(o))$ and is defined as follows:

(Def.9) For every natural number n such that $n \in \mathrm{dom}\,\mathrm{Arity}(o)$ holds $(R\#x)(n) = [x(n)]_{R(\pi_n \mathrm{Arity}(o))}$.

Let us consider S, o, let A be a non-empty algebra over S, and let R be a congruence of A. The functor $\mathrm{QuotRes}(R, o)$ yielding a function from ((the sorts of A) $\cdot$ (the result sort of S))(o) into ($\mathrm{Classes}\, R$ $\cdot$ (the result sort of S))(o) is defined as follows:

(Def.10) For every element x of (the sorts of A)(the result sort of o) holds $(\mathrm{QuotRes}(R, o))(x) = [x]_R$.

The functor $\mathrm{QuotArgs}(R, o)$ yielding a function from ((the sorts of $A)^{\#}$ $\cdot$ (the arity of S))(o) into (($\mathrm{Classes}\, R)^{\#}$ $\cdot$ (the arity of S))(o) is defined as follows:

(Def.11) For every element x of $\mathrm{Args}(o, A)$ holds $(\mathrm{QuotArgs}(R, o))(x) = R\#x$.

Let us consider S, let A be a non-empty algebra over S, and let R be a congruence of A. The functor $\mathrm{QuotRes}(R)$ yielding a many sorted function from (the sorts of A) $\cdot$ (the result sort of S) into $\mathrm{Classes}\, R$ $\cdot$ (the result sort of S) is defined as follows:

(Def.12) For every operation symbol o of S holds $(\mathrm{QuotRes}(R))(o) = \mathrm{QuotRes}(R, o)$.

The functor $\mathrm{QuotArgs}(R)$ yielding a many sorted function from (the sorts of $A)^{\#}$ $\cdot$ (the arity of S) into ($\mathrm{Classes}\, R)^{\#}$ $\cdot$ (the arity of S) is defined as follows:

(Def.13) For every operation symbol o of S holds $(\mathrm{QuotArgs}(R))(o) = \mathrm{QuotArgs}(R, o)$.

Next we state the proposition

(2) Let A be a non-empty algebra over S, and let R be a congruence of A, and let x be arbitrary. Suppose $x \in$ (($\mathrm{Classes}\, R)^{\#}$ $\cdot$ (the arity of S))(o). Then there exists an element a of $\mathrm{Args}(o, A)$ such that $x = R\#a$.

Let us consider S, o, let A be a non-empty algebra over S, and let R be a congruence of A. The functor $\mathrm{QuotCharact}(R, o)$ yields a function from (($\mathrm{Classes}\, R)^{\#}$ $\cdot$ (the arity of S))(o) into ($\mathrm{Classes}\, R$ $\cdot$ (the result sort of S))(o) and is defined as follows:

(Def.14) For every element a of $\mathrm{Args}(o, A)$ such that $R\#a \in ((\mathrm{Classes}\, R)^{\#} \cdot (\text{the arity of } S))(o)$ holds $(\mathrm{QuotCharact}(R, o))(R\#a) = (\mathrm{QuotRes}(R, o) \cdot \mathrm{Den}(o, A))(a)$.

Let us consider S, let A be a non-empty algebra over S, and let R be a congruence of A. The functor $\mathrm{QuotCharact}(R)$ yielding a many sorted function from $(\mathrm{Classes}\, R)^{\#} \cdot (\text{the arity of } S)$ into $\mathrm{Classes}\, R \cdot (\text{the result sort of } S)$ is defined as follows:

(Def.15) For every operation symbol o of S holds $(\mathrm{QuotCharact}(R))(o) = \mathrm{QuotCharact}(R, o)$.

Let us consider S, let U_1 be a non-empty algebra over S, and let R be a congruence of U_1. The functor $\mathrm{QuotMSAlg}(R)$ yielding a strict non-empty algebra over S is defined by:

(Def.16) $\mathrm{QuotMSAlg}(R) = \langle \mathrm{Classes}\, R, \mathrm{QuotCharact}(R) \rangle$.

Let us consider S, let U_1 be a non-empty algebra over S, let R be a congruence of U_1, and let s be a sort symbol of S. The functor $\mathrm{MSNatHom}(U_1, R, s)$ yielding a function from (the sorts of U_1)(s) into $(\mathrm{Classes}\, R)(s)$ is defined as follows:

(Def.17) For arbitrary x such that $x \in$ (the sorts of U_1)(s) holds $(\mathrm{MSNatHom}(U_1, R, s))(x) = [x]_{R(s)}$.

Let us consider S, let U_1 be a non-empty algebra over S, and let R be a congruence of U_1. The functor $\mathrm{MSNatHom}(U_1, R)$ yielding a many sorted function from U_1 into $\mathrm{QuotMSAlg}(R)$ is defined by:

(Def.18) For every sort symbol s of S holds $(\mathrm{MSNatHom}(U_1, R))(s) = \mathrm{MSNatHom}(U_1, R, s)$.

Next we state the proposition

(3) Let S be a non void non empty many sorted signature, and let U_1 be a non-empty algebra over S, and let R be a congruence of U_1. Then $\mathrm{MSNatHom}(U_1, R)$ is an epimorphism of U_1 onto $\mathrm{QuotMSAlg}(R)$.

Let us consider S, let U_1, U_2 be non-empty algebras over S, let F be a many sorted function from U_1 into U_2, and let s be a sort symbol of S. The functor $\mathrm{Congruence}(F, s)$ yields an equivalence relation of (the sorts of U_1)(s) and is defined as follows:

(Def.19) For all elements x, y of (the sorts of U_1)(s) holds $\langle x, y \rangle \in \mathrm{Congruence}(F, s)$ iff $F(s)(x) = F(s)(y)$.

Let us consider S, let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. Let us assume that F is a homomorphism of U_1 into U_2. The functor $\mathrm{Congruence}(F)$ yielding a congruence of U_1 is defined by:

(Def.20) For every sort symbol s of S holds $(\mathrm{Congruence}(F))(s) = \mathrm{Congruence}(F, s)$.

Let us consider S, let U_1, U_2 be non-empty algebras over S, let F be a many sorted function from U_1 into U_2, and let s be a sort symbol of S. Let us assume that F is a homomorphism of U_1 into U_2. The functor $\mathrm{MSHomQuot}(F, s)$ yields a function from (the sorts of QuotMSAlg(Congruence(F)))(s) into (the sorts of U_2)(s) and is defined as follows:

(Def.21) For every element x of (the sorts of U_1)(s) holds (MSHomQuot(F, s))($[x]_{\mathrm{Congruence}(F,s)}$) = $F(s)(x)$.

Let us consider S, let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. Let us assume that F is a homomorphism of U_1 into U_2. The functor MSHomQuot(F) yields a many sorted function from QuotMSAlg(Congruence(F)) into U_2 and is defined by:

(Def.22) For every sort symbol s of S holds (MSHomQuot(F))(s) = MSHomQuot(F, s).

The following propositions are true:

(4) Let S be a non void non empty many sorted signature, and let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then MSHomQuot(F) is a monomorphism of QuotMSAlg(Congruence(F)) into U_2.

(5) Let S be a non void non empty many sorted signature, and let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. Suppose F is an epimorphism of U_1 onto U_2. Then MSHomQuot(F) is an isomorphism of QuotMSAlg(Congruence(F)) and U_2.

(6) Let S be a non void non empty many sorted signature, and let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. If F is an epimorphism of U_1 onto U_2, then QuotMSAlg(Congruence(F)) and U_2 are isomorphic.

References

[1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Formalized Mathematics*, 5(**1**):47–54, 1996.
[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Mał gorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Formalized Mathematics*, 5(**1**):61–65, 1996.
[8] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.
[9] Beata Madras. Products of many sorted algebras. *Formalized Mathematics*, 5(**1**):55–60, 1996.
[10] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.
[11] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.
[12] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.

[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[14] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[15] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[16] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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Quantales

Grzegorz Bancerek
Institute of Mathematics
Polish Academy of Sciences

Summary. The concepts of Girard quantales (see [10] and [15]) and Blikle nets (see [5]) are introduced.

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The notation and terminology used in this paper are introduced in the following papers: [12], [11], [14], [7], [8], [6], [9], [16], [2], [3], [1], [13], and [4].

Let X be a set and let Y be a subset of 2^X. Then $\bigcup Y$ is a subset of X.

In this article we present several logical schemes. The scheme *DenestFraenkel* concerns a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding arbitrary, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(a) : a$ ranges over elements of $\mathcal{B}$, $a \in \{\mathcal{G}(b) : b$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[b]\}\} = \{\mathcal{F}(\mathcal{G}(a)) : a$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[a]\}$

for all values of the parameters.

The scheme *EmptyFraenkel* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding arbitrary, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(a) : a$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[a]\} = \emptyset$

provided the following requirement is met:

- It is not true that there exists an element a of $\mathcal{A}$ such that $\mathcal{P}[a]$.

We now state two propositions:

(1) Let L_1, L_2 be non empty lattice structures. Suppose the lattice structure of L_1 = the lattice structure of L_2. Let a_1, b_1 be elements of L_1, and let a_2, b_2 be elements of L_2, and let X be a set. Suppose $a_1 = a_2$ and $b_1 = b_2$. Then $a_1 \sqcup b_1 = a_2 \sqcup b_2$ and $a_1 \sqcap b_1 = a_2 \sqcap b_2$ and $a_1 \sqsubseteq b_1$ iff $a_2 \sqsubseteq b_2$.

(2) Let L_1, L_2 be non empty lattice structures. Suppose the lattice structure of L_1 = the lattice structure of L_2. Let a be an element of L_1, and

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let b be an element of L_2, and let X be a set. If $a = b$, then $a \sqsubseteq X$ iff $b \sqsubseteq X$ and $a \sqsupseteq X$ iff $b \sqsupseteq X$.

Let L be a 1-sorted structure. A binary operation on L is a binary operation on the carrier of L. A unary operation on L is a unary operation on the carrier of L.

Let L be a non empty lattice structure and let X be a subset of L. We say that X is directed if and only if:

(Def.1) For every finite subset Y of X there exists an element x of L such that $\bigsqcup_L Y \sqsubseteq x$ and $x \in X$.

The following proposition is true

(3) For every non empty lattice structure L and for every subset X of L such that X is directed holds X is non empty.

We introduce quantale structures which are extensions of lattice structure and half group structure and are systems

$\langle$ a carrier, a join operation, a meet operation, a multiplication $\rangle$,

where the carrier is a set and the join operation, the meet operation, and the multiplication are binary operations on the carrier.

Let us mention that there exists a quantale structure which is non empty.

We consider quasinet structures as extensions of quantale structure and multiplicative loop structure as systems

$\langle$ a carrier, a join operation, a meet operation, a multiplication, a unity $\rangle$,

where the carrier is a set, the join operation, the meet operation, and the multiplication are binary operations on the carrier, and the unity is an element of the carrier.

Let us note that there exists a quasinet structure which is non empty.

A non empty half group structure has left-zero if:

(Def.2) There exists an element a of it such that for every element b of it holds $a \cdot b = a$.

A non empty half group structure has right-zero if:

(Def.3) There exists an element b of it such that for every element a of it holds $a \cdot b = b$.

A non empty half group structure has zero if:

(Def.4) It has left-zero and right-zero.

One can verify that every non empty half group structure which has zero has also left-zero and right-zero and every non empty half group structure which has left-zero and right-zero has also zero.

Let us note that there exists a non empty half group structure has zero.

A non empty quantale structure is right-distributive if:

(Def.5) For every element a of it and for every set X holds $a \otimes \bigsqcup_{\rm it} X = \bigsqcup_{\rm it}\{a \otimes b :$ b ranges over elements of it, $b \in X\}$.

A non empty quantale structure is left-distributive if:

(Def.6) For every element a of it and for every set X holds $\bigsqcup_{\rm it} X \otimes a = \bigsqcup_{\rm it}\{b \otimes a :$ b ranges over elements of it, $b \in X\}$.

A non empty quantale structure is $\otimes$-additive if:

(Def.7) For all elements a, b, c of it holds $(a \sqcup b) \otimes c = a \otimes c \sqcup b \otimes c$ and $c \otimes (a \sqcup b) = c \otimes a \sqcup c \otimes b$.

A non empty quantale structure is $\otimes$-continuous if it satisfies the condition (Def.8).

(Def.8) Let X_1, X_2 be subsets of it. Suppose X_1 is directed and X_2 is directed. Then $\bigsqcup X_1 \otimes \bigsqcup X_2 = \bigsqcup_{\rm it}\{a \otimes b : a$ ranges over elements of it, b ranges over elements of it, $a \in X_1 \ \wedge \ b \in X_2\}$.

The following proposition is true

(4) Let Q be a non empty quantale structure. Suppose the lattice structure of $Q =$ the lattice of subsets of $\emptyset$. Then Q is associative commutative unital complete right-distributive left-distributive and lattice-like and has zero.

Let A be a non empty set and let b_1, b_2, b_3 be binary operations on A. Note that $\langle A, b_1, b_2, b_3 \rangle$ is non empty.

Let us observe that there exists a non empty quantale structure which is associative commutative unital left-distributive right-distributive complete and lattice-like and has zero.

The scheme *LUBFraenkelDistr* deals with a complete lattice-like non empty quantale structure $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and sets $\mathcal{B}$, $\mathcal{C}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{F}(a,b) : b$ ranges over elements of $\mathcal{A}$, $b \in \mathcal{C}\} : a$ ranges over elements of $\mathcal{A}$, $a \in \mathcal{B}\} = \bigsqcup_{\mathcal{A}}\{\mathcal{F}(a,b) : a$ ranges over elements of $\mathcal{A}$, b ranges over elements of $\mathcal{A}$, $a \in \mathcal{B} \ \wedge \ b \in \mathcal{C}\}$

for all values of the parameters.

In the sequel Q denotes a left-distributive right-distributive complete lattice-like non empty quantale structure and a, b, c denote elements of Q.

Next we state two propositions:

(5) For every Q and for all sets X, Y holds $\bigsqcup_Q X \otimes \bigsqcup_Q Y = \bigsqcup_Q\{a \otimes b : a \in X \ \wedge \ b \in Y\}$.

(6) $(a \sqcup b) \otimes c = a \otimes c \sqcup b \otimes c$ and $c \otimes (a \sqcup b) = c \otimes a \sqcup c \otimes b$.

Let A be a non empty set, let b_1, b_2, b_3 be binary operations on A, and let e be an element of A. Observe that $\langle A, b_1, b_2, b_3, e \rangle$ is non empty.

One can verify that there exists a non empty quasinet structure which is complete and lattice-like.

Let us note that every complete lattice-like non empty quasinet structure which is left-distributive and right-distributive is also $\otimes$-continuous and $\otimes$-additive.

Let us observe that there exists a non empty quasinet structure which is associative commutative well unital left-distributive right-distributive complete and lattice-like and has zero and left-zero.

A quantale is an associative left-distributive right-distributive complete lattice-like non empty quantale structure. A quasinet is a well unital associative $\otimes$-continuous $\otimes$-additive complete lattice-like non empty quasinet structure with left-zero.

A Blikle net is a non empty quasinet with zero.

The following proposition is true

(7) For every well unital non empty quasinet structure Q such that Q is a quantale holds Q is a Blikle net.

We adopt the following rules: Q will be a quantale and a, b, c, d, D will be elements of Q.

The following propositions are true:

(8) If $a \sqsubseteq b$, then $a \otimes c \sqsubseteq b \otimes c$ and $c \otimes a \sqsubseteq c \otimes b$.

(9) If $a \sqsubseteq b$ and $c \sqsubseteq d$, then $a \otimes c \sqsubseteq b \otimes d$.

Let A be a non empty set. A unary operation on A is idempotent if:

(Def.9) For every element a of A holds $\mathrm{it}(\mathrm{it}(a)) = \mathrm{it}(a)$.

Let L be a non empty lattice structure. A unary operation on L is inflationary if:

(Def.10) For every element p of L holds $p \sqsubseteq \mathrm{it}(p)$.

A unary operation on L is deflationary if:

(Def.11) For every element p of L holds $\mathrm{it}(p) \sqsubseteq p$.

A unary operation on L is monotone if:

(Def.12) For all elements p, q of L such that $p \sqsubseteq q$ holds $\mathrm{it}(p) \sqsubseteq \mathrm{it}(q)$.

A unary operation on L is $\bigsqcup$-distributive if:

(Def.13) For every subset X of L holds $\mathrm{it}(\bigsqcup X) \sqsubseteq \bigsqcup_L \{\mathrm{it}(a) : a$ ranges over elements of L, $a \in X\}$.

We now state the proposition

(10) Let L be a complete lattice and let j be a unary operation on L. Suppose j is monotone. Then j is $\bigsqcup$-distributive if and only if for every subset X of L holds $j(\bigsqcup X) = \bigsqcup_L \{j(a) : a$ ranges over elements of L, $a \in X\}$.

Let Q be a non empty quantale structure. A unary operation on Q is $\otimes$-monotone if:

(Def.14) For all elements a, b of Q holds $\mathrm{it}(a) \otimes \mathrm{it}(b) \sqsubseteq \mathrm{it}(a \otimes b)$.

Let Q be a non empty quantale structure and let a, b be elements of Q. The functor $a \to_r b$ yields an element of Q and is defined by:

(Def.15) $a \to_r b = \bigsqcup_Q \{c : c$ ranges over elements of Q, $c \otimes a \sqsubseteq b\}$.

The functor $a \to_l b$ yields an element of Q and is defined by:

(Def.16) $a \to_l b = \bigsqcup_Q \{c : c$ ranges over elements of Q, $a \otimes c \sqsubseteq b\}$.

One can prove the following propositions:

(11) $a \otimes b \sqsubseteq c$ iff $b \sqsubseteq a \to_l c$.

(12) $a \otimes b \sqsubseteq c$ iff $a \sqsubseteq b \to_r c$.

(13) For every quantale Q and for all elements s, a, b of Q such that $a \sqsubseteq b$ holds $b \to_r s \sqsubseteq a \to_r s$ and $b \to_l s \sqsubseteq a \to_l s$.

(14) Let Q be a quantale, and let s be an element of Q, and let j be a unary operation on Q. If for every element a of Q holds $j(a) = (a \to_r s) \to_r s$, then j is monotone.

Let Q be a non empty quantale structure. An element of Q is dualizing if:

(Def.17) For every element a of Q holds $(a \to_r \text{it}) \to_l \text{it} = a$ and $(a \to_l \text{it}) \to_r \text{it} = a$.

An element of Q is cyclic if:

(Def.18) For every element a of Q holds $a \to_r \text{it} = a \to_l \text{it}$.

We now state several propositions:

(15) c is cyclic iff for all a, b such that $a \otimes b \sqsubseteq c$ holds $b \otimes a \sqsubseteq c$.

(16) For every quantale Q and for all elements s, a of Q such that s is cyclic holds $a \sqsubseteq (a \to_r s) \to_r s$ and $a \sqsubseteq (a \to_l s) \to_l s$.

(17) For every quantale Q and for all elements s, a of Q such that s is cyclic holds $a \to_r s = ((a \to_r s) \to_r s) \to_r s$ and $a \to_l s = ((a \to_l s) \to_l s) \to_l s$.

(18) For every quantale Q and for all elements s, a, b of Q such that s is cyclic holds $((a \to_r s) \to_r s) \otimes ((b \to_r s) \to_r s) \sqsubseteq (a \otimes b \to_r s) \to_r s$.

(19) If D is dualizing, then Q is unital and $\mathbf{1}_{\text{the multiplication of } Q} = D \to_r D$ and $\mathbf{1}_{\text{the multiplication of } Q} = D \to_l D$.

(20) If a is dualizing, then $b \to_r c = b \otimes (c \to_l a) \to_r a$ and $b \to_l c = (c \to_r a) \otimes b \to_l a$.

We introduce Girard quantale structures which are extensions of quasinet structure and are systems

⟨ a carrier, a join operation, a meet operation, a multiplication, a unity, absurd ⟩,

where the carrier is a set, the join operation, the meet operation, and the multiplication are binary operations on the carrier, and the unity and the absurd constitute elements of the carrier.

One can check that there exists a Girard quantale structure which is non empty.

A non empty Girard quantale structure is cyclic if:

(Def.19) The absurd of it is cyclic.

A non empty Girard quantale structure is dualized if:

(Def.20) The absurd of it is dualizing.

The following proposition is true

(21) Let Q be a non empty Girard quantale structure. Suppose the lattice structure of Q = the lattice of subsets of $\emptyset$. Then Q is cyclic and dualized.

Let A be a non empty set, let b_1, b_2, b_3 be binary operations on A, and let e_1, e_2 be elements of A. One can verify that $\langle A, b_1, b_2, b_3, e_1, e_2 \rangle$ is non empty.

Let us note that there exists a non empty Girard quantale structure which is associative commutative well unital left-distributive right-distributive complete lattice-like cyclic dualized and strict.

A Girard quantale is an associative well unital left-distributive right-distributive complete lattice-like cyclic dualized non empty Girard quantale structure.

Let G be a Girard quantale structure. The functor $\bot_G$ yielding an element of G is defined as follows:

(Def.21) $\bot_G =$ the absurd of G.

Let G be a non empty Girard quantale structure. The functor $\top_G$ yielding an element of G is defined by:

(Def.22) $\top_G = \bot_G \to_r \bot_G$.

Let a be an element of G. The functor $\bot_a$ yielding an element of G is defined by:

(Def.23) $\bot_a = a \to_r \bot_G$.

Let G be a non empty Girard quantale structure. The functor Negation(G) yields a unary operation on G and is defined as follows:

(Def.24) For every element a of G holds $(\mathrm{Negation}(G))(a) = \bot_a$.

Let G be a non empty Girard quantale structure and let u be a unary operation on G. The functor $\bot_u$ yielding a unary operation on G is defined by:

(Def.25) $\bot_u = \mathrm{Negation}(G) \cdot u$.

Let G be a non empty Girard quantale structure and let o be a binary operation on G. The functor $\bot_o$ yields a binary operation on G and is defined as follows:

(Def.26) $\bot_o = \mathrm{Negation}(G) \cdot o$.

We adopt the following convention: Q denotes a Girard quantale, a, a_1, a_2, b, b_1, b_2, c denote elements of Q, and X denotes a set.

We now state several propositions:

(22) $\bot_{\bot_a} = a$.

(23) If $a \sqsubseteq b$, then $\bot_b \sqsubseteq \bot_a$.

(24) $\bot_{\bigsqcup_Q X} = \bigsqcap_Q \{\bot_a : a \in X\}$.

(25) $\bot_{\bigsqcap_Q X} = \bigsqcup_Q \{\bot_a : a \in X\}$.

(26) $\bot_{a \sqcup b} = \bot_a \sqcap \bot_b$ and $\bot_{a \sqcap b} = \bot_a \sqcup \bot_b$.

Let us consider Q, a, b. The functor $a \wp b$ yields an element of Q and is defined as follows:

(Def.27) $a \wp b = \bot_{\bot_a \otimes \bot_b}$.

We now state several propositions:

(27) $a \otimes \bigsqcup_Q X = \bigsqcup_Q \{a \otimes b : b \in X\}$ and $a \wp \bigsqcap_Q X = \bigsqcap_Q \{a \wp c : c \in X\}$.

(28) $\bigsqcup_Q X \otimes a = \bigsqcup_Q \{b \otimes a : b \in X\}$ and $\bigsqcap_Q X \wp a = \bigsqcap_Q \{c \wp a : c \in X\}$.

(29) $a \wp b \sqcap c = (a \wp b) \sqcap (a \wp c)$ and $b \sqcap c \wp a = (b \wp a) \sqcap (c \wp a)$.

(30) If $a_1 \sqsubseteq b_1$ and $a_2 \sqsubseteq b_2$, then $a_1 \wp a_2 \sqsubseteq b_1 \wp b_2$.

(31) $(a \wp b) \wp c = a \wp (b \wp c)$.

(32) $a \otimes \top_Q = a$ and $\top_Q \otimes a = a$.

(33) $a \wp \bot_Q = a$ and $\bot_Q \wp a = a$.

(34) Let Q be a quantale and let j be a unary operation on Q. Suppose j is monotone idempotent and $\bigsqcup$-distributive. Then there exists a complete lattice L such that the carrier of $L = \operatorname{rng} j$ and for every subset X of L holds $\bigsqcup X = j(\bigsqcup_Q X)$.

References

[1] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.

[2] Grzegorz Bancerek. Filters - part I. *Formalized Mathematics*, 1(**5**):813–819, 1990.

[3] Grzegorz Bancerek. Filters - part II. Quotient lattices modulo filters and direct product of two lattices. *Formalized Mathematics*, 2(**3**):433–438, 1991.

[4] Grzegorz Bancerek. Monoids. *Formalized Mathematics*, 3(**2**):213–225, 1992.

[5] A. Blikle. An analysis of programs by algebraic means. *Banach Center Publications*, 2:167–213.

[6] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[10] J.-Y. Girard. Liniear logic. *Theoretical Computer Science*, 50(1):1–102, 1987.

[11] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[13] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.

[14] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[15] Davide N. Yetter. Quantales and (noncommutative) linear logic. *The Journal of Symbolic Logic*, 55(1):41–64, March 1990.

[16] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

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Sequences in $\mathcal{E}^N_{\rm T}$

Agnieszka Sakowicz
Warsaw University
Białystok

Jarosław Gryko
Warsaw University
Białystok

Adam Grabowski
Warsaw University
Białystok

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The papers [12], [3], [4], [11], [8], [10], [1], [2], [5], [6], [9], and [7] provide the notation and terminology for this paper.

For simplicity we adopt the following rules: f denotes a function, N, n, m denote natural numbers, q, r, r_1, r_2 denote real numbers, x is arbitrary, and w, w_1, w_2, g denote points of $\mathcal{E}^N_{\rm T}$.

Let us consider N. A sequence in $\mathcal{E}^N_{\rm T}$ is a function from $\mathbb{N}$ into the carrier of $\mathcal{E}^N_{\rm T}$.

In the sequel s_1, s_2, s_3, s_4, s'_1 are sequences in $\mathcal{E}^N_{\rm T}$.

Next we state two propositions:

(1) f is a sequence in $\mathcal{E}^N_{\rm T}$ if and only if $\operatorname{dom} f = \mathbb{N}$ and for every x such that $x \in \mathbb{N}$ holds $f(x)$ is a point of $\mathcal{E}^N_{\rm T}$.

(2) f is a sequence in $\mathcal{E}^N_{\rm T}$ iff $\operatorname{dom} f = \mathbb{N}$ and for every n holds $f(n)$ is a point of $\mathcal{E}^N_{\rm T}$.

Let us consider N, s_1, n. Then $s_1(n)$ is a point of $\mathcal{E}^N_{\rm T}$.

Let us consider N. A sequence in $\mathcal{E}^N_{\rm T}$ is non-zero if:

(Def.1) $\operatorname{rng} \mathrm{it} \subseteq (\text{the carrier of } \mathcal{E}^N_{\rm T}) \setminus \{0_{\mathcal{E}^N_{\rm T}}\}$.

We now state several propositions:

(3) s_1 is non-zero iff for every x such that $x \in \mathbb{N}$ holds $s_1(x) \neq 0_{\mathcal{E}^N_{\rm T}}$.

(4) s_1 is non-zero iff for every n holds $s_1(n) \neq 0_{\mathcal{E}^N_{\rm T}}$.

(5) For all N, s_1, s_2 such that for every x such that $x \in \mathbb{N}$ holds $s_1(x) = s_2(x)$ holds $s_1 = s_2$.

(6) For all N, s_1, s_2 such that for every n holds $s_1(n) = s_2(n)$ holds $s_1 = s_2$.

(7) For every point w of $\mathcal{E}^N_{\rm T}$ there exists s_1 such that $\operatorname{rng} s_1 = \{w\}$.

The scheme *ExTopRealNSeq* deals with a natural number $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{E}^{\mathcal{A}}_{\rm T}$, and states that:

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There exists a sequence s_1 in $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ such that for every n holds $s_1(n) = \mathcal{F}(n)$

for all values of the parameters.

Let us consider N, s_2, s_3. The functor $s_2 + s_3$ yielding a sequence in $\mathcal{E}^{N}_{\mathrm{T}}$ is defined by:

(Def.2) For every n holds $(s_2 + s_3)(n) = s_2(n) + s_3(n)$.

Let us consider r, N, s_1. The functor $r \cdot s_1$ yields a sequence in $\mathcal{E}^{N}_{\mathrm{T}}$ and is defined by:

(Def.3) For every n holds $(r \cdot s_1)(n) = r \cdot s_1(n)$.

Let us consider N, s_1. The functor $-s_1$ yields a sequence in $\mathcal{E}^{N}_{\mathrm{T}}$ and is defined as follows:

(Def.4) For every n holds $(-s_1)(n) = -s_1(n)$.

Let us consider N, s_2, s_3. The functor $s_2 - s_3$ yields a sequence in $\mathcal{E}^{N}_{\mathrm{T}}$ and is defined by:

(Def.5) $s_2 - s_3 = s_2 + -s_3$.

Let us consider N and let x be a point of $\mathcal{E}^{N}_{\mathrm{T}}$. The functor $|x|$ yields a real number and is defined by:

(Def.6) There exists a finite sequence y of elements of $\mathbb{R}$ such that $x = y$ and $|x| = |y|$.

Let us consider N, s_1. The functor $|s_1|$ yielding a sequence of real numbers is defined by:

(Def.7) For every n holds $|s_1|(n) = |s_1(n)|$.

We now state a number of propositions:

(8) $|r| \cdot |w| = |r \cdot w|$.

(9) $|r \cdot s_1| = |r|\,|s_1|$.

(10) $s_2 + s_3 = s_3 + s_2$.

(11) $(s_2 + s_3) + s_4 = s_2 + (s_3 + s_4)$.

(12) $-s_1 = (-1) \cdot s_1$.

(13) $r \cdot (s_2 + s_3) = r \cdot s_2 + r \cdot s_3$.

(14) $(r \cdot q) \cdot s_1 = r \cdot (q \cdot s_1)$.

(15) $r \cdot (s_2 - s_3) = r \cdot s_2 - r \cdot s_3$.

(16) $s_2 - (s_3 + s_4) = s_2 - s_3 - s_4$.

(17) $1 \cdot s_1 = s_1$.

(18) $--s_1 = s_1$.

(19) $s_2 - -s_3 = s_2 + s_3$.

(20) $s_2 - (s_3 - s_4) = (s_2 - s_3) + s_4$.

(21) $s_2 + (s_3 - s_4) = (s_2 + s_3) - s_4$.

(22) If $r \neq 0$ and s_1 is non-zero, then $r \cdot s_1$ is non-zero.

(23) If s_1 is non-zero, then $-s_1$ is non-zero.

(24) $|0_{\mathcal{E}^{N}_{\mathrm{T}}}| = 0$.

(25) If $|w| = 0$, then $w = 0_{\mathcal{E}^N_{\mathrm{T}}}$.

(26) $|w| \geq 0$.

(27) $|-w| = |w|$.

(28) $|w_1 - w_2| = |w_2 - w_1|$.

(29) $|w_1 - w_2| = 0$ iff $w_1 = w_2$.

(30) $|w_1 + w_2| \leq |w_1| + |w_2|$.

(31) $|w_1 - w_2| \leq |w_1| + |w_2|$.

(32) $|w_1| - |w_2| \leq |w_1 + w_2|$.

(33) $|w_1| - |w_2| \leq |w_1 - w_2|$.

(34) If $w_1 \neq w_2$, then $|w_1 - w_2| > 0$.

(35) $|w_1 - w_2| \leq |w_1 - w| + |w - w_2|$.

(36) If $0 \leq |w_1|$ and $0 \leq r_1$ and $|w_1| < |w_2|$ and $r_1 < r_2$, then $|w_1| \cdot r_1 < |w_2| \cdot r_2$.

(38)[1] $-|w| < r$ and $r < |w|$ iff $|r| < |w|$.

Let us consider N. A sequence in $\mathcal{E}^N_{\mathrm{T}}$ is bounded if:

(Def.8) There exists r such that for every n holds $|\mathrm{it}(n)| < r$.

The following proposition is true

(39) For every n there exists r such that $0 < r$ and for every m such that $m \leq n$ holds $|s_1(m)| < r$.

Let us consider N. A sequence in $\mathcal{E}^N_{\mathrm{T}}$ is convergent if:

(Def.9) There exists g such that for every r such that $0 < r$ there exists n such that for every m such that $n \leq m$ holds $|\mathrm{it}(m) - g| < r$.

Let us consider N, s_1. Let us assume that s_1 is convergent. The functor $\lim s_1$ yields a point of $\mathcal{E}^N_{\mathrm{T}}$ and is defined by:

(Def.10) For every r such that $0 < r$ there exists n such that for every m such that $n \leq m$ holds $|s_1(m) - \lim s_1| < r$.

The following propositions are true:

(40) Suppose s_1 is convergent. Then $\lim s_1 = g$ if and only if for every r such that $0 < r$ there exists n such that for every m such that $n \leq m$ holds $|s_1(m) - g| < r$.

(41) If s_1 is convergent and s'_1 is convergent, then $s_1 + s'_1$ is convergent.

(42) If s_1 is convergent and s'_1 is convergent, then $\lim(s_1 + s'_1) = \lim s_1 + \lim s'_1$.

(43) If s_1 is convergent, then $r \cdot s_1$ is convergent.

(44) If s_1 is convergent, then $\lim(r \cdot s_1) = r \cdot \lim s_1$.

(45) If s_1 is convergent, then $-s_1$ is convergent.

(46) If s_1 is convergent, then $\lim(-s_1) = -\lim s_1$.

(47) If s_1 is convergent and s'_1 is convergent, then $s_1 - s'_1$ is convergent.

[1]The proposition (37) has been removed.

(48) If s_1 is convergent and s'_1 is convergent, then $\lim(s_1 - s'_1) = \lim s_1 - \lim s'_1$.

(50)[2] If s_1 is convergent, then s_1 is bounded.

(51) If s_1 is convergent, then if $\lim s_1 \neq 0_{\mathcal{E}^N_{\rm T}}$, then there exists n such that for every m such that $n \leq m$ holds $\frac{|\lim s_1|}{2} < |s_1(m)|$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.

[6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[8] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[9] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[10] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[12] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

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[2]The proposition (49) has been removed.

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Extremal Properties of Vertices on Special Polygons, Part I

Yatsuka Nakamura
Shinshu University
Nagano

Czesław Byliński
Warsaw University
Białystok

Summary. First, extremal properties of endpoints of line segments in n-dimensional Euclidean space are discussed. Some topological properties of line segments are also discussed. Secondly, extremal properties of vertices of special polygons which consist of horizontal and vertical line segments in 2-dimensional Euclidean space, are also derived.

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The terminology and notation used in this paper are introduced in the following articles: [18], [2], [12], [17], [21], [19], [22], [6], [15], [10], [16], [1], [7], [3], [5], [13], [4], [8], [20], [9], [14], and [11].

1. Preliminaries

One can prove the following propositions:

(1) For every finite sequence f holds f is trivial iff $\operatorname{len} f < 2$.

(2) For every finite set A holds A is trivial iff $\operatorname{card} A < 2$.

(3) For every set A holds A is non trivial iff there exist arbitrary a_1, a_2 such that $a_1 \in A$ and $a_2 \in A$ and $a_1 \neq a_2$.

(4) Let D be a non empty set and let A be a subset of D. Then A is non trivial if and only if there exist elements d_1, d_2 of D such that $d_1 \in A$ and $d_2 \in A$ and $d_1 \neq d_2$.

We follow a convention: n, i, k, m will denote natural numbers and r, r_1, r_2, s, s_1, s_2 will denote real numbers.

Next we state a number of propositions:

(5) If $n \leq k$, then $n - 1 \leq k$ and $n - 1 < k$ and $n \leq k + 1$ and $n < k + 1$.

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(6) If $n < k$, then $n - 1 \leq k$ and $n - 1 < k$ and $n + 1 \leq k$ and $n \leq k - 1$ and $n \leq k + 1$ and $n < k + 1$.

(7) If $1 \leq k - m$ and $k - m \leq n$, then $k - m \in \mathrm{Seg}\, n$ and $k - m$ is a natural number.

(8) If $r_1 \geq 0$ and $r_2 \geq 0$ and $r_1 + r_2 = 0$, then $r_1 = 0$ and $r_2 = 0$.

(9) If $r_1 \leq 0$ and $r_2 \leq 0$ and $r_1 + r_2 = 0$, then $r_1 = 0$ and $r_2 = 0$.

(10) If $0 \leq r_1$ and $r_1 \leq 1$ and $0 \leq r_2$ and $r_2 \leq 1$ and $r_1 \cdot r_2 = 1$, then $r_1 = 1$ and $r_2 = 1$.

(11) If $r_1 \geq 0$ and $r_2 \geq 0$ and $s_1 \geq 0$ and $s_2 \geq 0$ and $r_1 \cdot s_1 + r_2 \cdot s_2 = 0$, then $r_1 = 0$ or $s_1 = 0$ but $r_2 = 0$ or $s_2 = 0$.

(12) If $0 \leq r$ and $r \leq 1$ and $s_1 \geq 0$ and $s_2 \geq 0$ and $r \cdot s_1 + (1 - r) \cdot s_2 = 0$, then $r = 0$ and $s_2 = 0$ or $r = 1$ and $s_1 = 0$ or $s_1 = 0$ and $s_2 = 0$.

(13) If $r < r_1$ and $r < r_2$, then $r < \min(r_1, r_2)$.

(14) If $r > r_1$ and $r > r_2$, then $r > \max(r_1, r_2)$.

In this article we present several logical schemes. The scheme *FinSeqFam* deals with a non empty set $\mathcal{A}$, a finite sequence $\mathcal{B}$ of elements of $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(\mathcal{B}, i) : i \in \mathrm{dom}\, \mathcal{B} \ \wedge \ \mathcal{P}[i]\}$ is finite

for all values of the parameters.

The scheme *FinSeqFam'* concerns a non empty set $\mathcal{A}$, a finite sequence $\mathcal{B}$ of elements of $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(\mathcal{B}, i) : 1 \leq i \ \wedge \ i \leq \mathrm{len}\, \mathcal{B} \ \wedge \ \mathcal{P}[i]\}$ is finite

for all values of the parameters.

Next we state several propositions:

(15) For all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|x_1 - x_2| - |x_2 - x_3| \leq |x_1 - x_3|$.

(16) For all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|x_2 - x_1| - |x_2 - x_3| \leq |x_3 - x_1|$.

(17) Every point of $\mathcal{E}^n_{\mathrm{T}}$ is an element of $\mathcal{R}^n$ and a point of $\mathcal{E}^n$.

(18) Every point of $\mathcal{E}^n$ is an element of $\mathcal{R}^n$ and a point of $\mathcal{E}^n_{\mathrm{T}}$.

(19) Every element of $\mathcal{R}^n$ is a point of $\mathcal{E}^n$ and a point of $\mathcal{E}^n_{\mathrm{T}}$.

2. Properties of line segments

In the sequel p, p_1, p_2, q_1, q_2 will denote points of $\mathcal{E}^n_{\mathrm{T}}$.

One can prove the following propositions:

(20) For all points u_1, u_2 of $\mathcal{E}^n$ and for all elements v_1, v_2 of $\mathcal{R}^n$ such that $v_1 = u_1$ and $v_2 = u_2$ holds $\rho(u_1, u_2) = |v_1 - v_2|$.

(21) For all p, p_1, p_2 such that $p \in \mathcal{L}(p_1, p_2)$ there exists r such that $0 \leq r$ and $r \leq 1$ and $p = (1 - r) \cdot p_1 + r \cdot p_2$.

(22) For all p_1, p_2, r such that $0 \leq r$ and $r \leq 1$ holds $(1 - r) \cdot p_1 + r \cdot p_2 \in \mathcal{L}(p_1, p_2)$.

(23) Given p_1, p_2 and let P be a non empty subset of $\mathcal{E}^n_{\rm T}$. Suppose P is closed and $P \subseteq \mathcal{L}(p_1, p_2)$. Then there exists s such that $(1 - s) \cdot p_1 + s \cdot p_2 \in P$ and for every r such that $0 \leq r$ and $r \leq 1$ and $(1 - r) \cdot p_1 + r \cdot p_2 \in P$ holds $s \leq r$.

(24) For all p_1, p_2, q_1, q_2 such that $\mathcal{L}(q_1, q_2) \subseteq \mathcal{L}(p_1, p_2)$ and $p_1 \in \mathcal{L}(q_1, q_2)$ holds $p_1 = q_1$ or $p_1 = q_2$.

(25) For all p_1, p_2, q_1, q_2 such that $\mathcal{L}(p_1, p_2) = \mathcal{L}(q_1, q_2)$ holds $p_1 = q_1$ and $p_2 = q_2$ or $p_1 = q_2$ and $p_2 = q_1$.

(26) $\mathcal{E}^n_{\rm T}$ is a $\rm T_2$ space.

(27) $\{p\}$ is closed.

(28) $\mathcal{L}(p_1, p_2)$ is compact.

(29) $\mathcal{L}(p_1, p_2)$ is closed.

Let us consider n, p and let P be a subset of $\mathcal{E}^n_{\rm T}$. We say that p is extremal in P if and only if:

(Def.1) $p \in P$ and for all p_1, p_2 such that $p \in \mathcal{L}(p_1, p_2)$ and $\mathcal{L}(p_1, p_2) \subseteq P$ holds $p = p_1$ or $p = p_2$.

We now state several propositions:

(30) For all subsets P, Q of $\mathcal{E}^n_{\rm T}$ such that p is extremal in P and $Q \subseteq P$ and $p \in Q$ holds p is extremal in Q.

(31) p is extremal in $\{p\}$.

(32) p_1 is extremal in $\mathcal{L}(p_1, p_2)$.

(33) p_2 is extremal in $\mathcal{L}(p_1, p_2)$.

(34) If p is extremal in $\mathcal{L}(p_1, p_2)$, then $p = p_1$ or $p = p_2$.

3. Alternating special sequences

We follow the rules: P, Q will be subsets of $\mathcal{E}^2_{\rm T}$, f, f_1, f_2 will be finite sequences of elements of the carrier of $\mathcal{E}^2_{\rm T}$, and p, p_1, p_2, p_3, q will be points of $\mathcal{E}^2_{\rm T}$.

The following proposition is true

(35) For all p_1, p_2 such that $(p_1)_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$ there exists p such that $p \in \mathcal{L}(p_1, p_2)$ and $p_{\mathbf{1}} \neq (p_1)_{\mathbf{1}}$ and $p_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq (p_1)_{\mathbf{2}}$ and $p_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$.

Let us consider P. We say that P is horizontal if and only if:

(Def.2) For all p, q such that $p \in P$ and $q \in P$ holds $p_{\mathbf{2}} = q_{\mathbf{2}}$.

We say that P is vertical if and only if:

(Def.3) For all p, q such that $p \in P$ and $q \in P$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$.

Let us observe that every subset of $\mathcal{E}^2_{\rm T}$ which is non trivial and horizontal is also non vertical and every subset of $\mathcal{E}^2_{\rm T}$ which is non trivial and vertical is also non horizontal.

Next we state a number of propositions:

(36) $p_{\mathbf{2}} = q_{\mathbf{2}}$ iff $\mathcal{L}(p,q)$ is horizontal.

(37) $p_{\mathbf{1}} = q_{\mathbf{1}}$ iff $\mathcal{L}(p,q)$ is vertical.

(38) If $p_1 \in \mathcal{L}(p,q)$ and $p_2 \in \mathcal{L}(p,q)$ and $(p_1)_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$, then $\mathcal{L}(p,q)$ is horizontal.

(39) If $p_1 \in \mathcal{L}(p,q)$ and $p_2 \in \mathcal{L}(p,q)$ and $(p_1)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$ and $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$, then $\mathcal{L}(p,q)$ is vertical.

(40) $\mathcal{L}(f,i)$ is closed.

(41) If f is special, then $\mathcal{L}(f,i)$ is vertical or $\mathcal{L}(f,i)$ is horizontal.

(42) If f is one-to-one and $1 \leq i$ and $i+1 \leq \operatorname{len} f$, then $\mathcal{L}(f,i)$ is non trivial.

(43) If f is one-to-one and $1 \leq i$ and $i+1 \leq \operatorname{len} f$ and $\mathcal{L}(f,i)$ is vertical, then $\mathcal{L}(f,i)$ is non horizontal.

(44) For every f holds $\{\mathcal{L}(f,i) : 1 \leq i \ \wedge \ i \leq \operatorname{len} f\}$ is finite.

(45) For every f holds $\{\mathcal{L}(f,i) : 1 \leq i \ \wedge \ i+1 \leq \operatorname{len} f\}$ is finite.

(46) For every f holds $\{\mathcal{L}(f,i) : 1 \leq i \ \wedge \ i \leq \operatorname{len} f\}$ is a family of subsets of $\mathcal{E}^2_{\mathrm{T}}$.

(47) For every f holds $\{\mathcal{L}(f,i) : 1 \leq i \ \wedge \ i+1 \leq \operatorname{len} f\}$ is a family of subsets of $\mathcal{E}^2_{\mathrm{T}}$.

(48) For every f such that $Q = \bigcup\{\mathcal{L}(f,i) : 1 \leq i \ \wedge \ i+1 \leq \operatorname{len} f\}$ holds Q is closed.

(49) $\widetilde{\mathcal{L}}(f)$ is closed.

A finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is alternating if:

(Def.4) For every i such that $1 \leq i$ and $i+2 \leq \operatorname{len} \mathrm{it}$ holds $(\pi_i \mathrm{it})_{\mathbf{1}} \neq (\pi_{i+2} \mathrm{it})_{\mathbf{1}}$ and $(\pi_i \mathrm{it})_{\mathbf{2}} \neq (\pi_{i+2} \mathrm{it})_{\mathbf{2}}$.

One can prove the following propositions:

(50) If f is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $(\pi_i f)_{\mathbf{1}} = (\pi_{i+1} f)_{\mathbf{1}}$, then $(\pi_{i+1} f)_{\mathbf{2}} = (\pi_{i+2} f)_{\mathbf{2}}$.

(51) If f is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $(\pi_i f)_{\mathbf{2}} = (\pi_{i+1} f)_{\mathbf{2}}$, then $(\pi_{i+1} f)_{\mathbf{1}} = (\pi_{i+2} f)_{\mathbf{1}}$.

(52) Suppose f is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $p_1 = \pi_i f$ and $p_2 = \pi_{i+1} f$ and $p_3 = \pi_{i+2} f$. Then $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$.

(53) Suppose f is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $p_1 = \pi_i f$ and $p_2 = \pi_{i+1} f$ and $p_3 = \pi_{i+2} f$. Then $(p_2)_{\mathbf{1}} = (p_3)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ or $(p_2)_{\mathbf{2}} = (p_3)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$.

(54) If f is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$, then $\mathcal{L}(\pi_i f, \pi_{i+2} f) \not\subseteq \mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$.

(55) If f is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $\mathcal{L}(f,i)$ is vertical, then $\mathcal{L}(f,i+1)$ is horizontal.

(56) If f is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $\mathcal{L}(f,i)$ is horizontal, then $\mathcal{L}(f,i+1)$ is vertical.

(57) Suppose f is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$. Then $\mathcal{L}(f,i)$ is vertical and $\mathcal{L}(f,i+1)$ is horizontal or $\mathcal{L}(f,i)$ is horizontal and $\mathcal{L}(f,i+1)$ is vertical.

(58) Suppose f is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $\pi_{i+1} f \in \mathcal{L}(p,q)$ and $\mathcal{L}(p,q) \subseteq \mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$. Then $\pi_{i+1} f = p$ or $\pi_{i+1} f = q$.

(59) If f is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$, then $\pi_{i+1} f$ is extremal in $\mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$.

(60) Let u be a point of $\mathcal{E}^2$. Suppose f is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $u = \pi_{i+1} f$ and $\pi_{i+1} f \in \mathcal{L}(p,q)$ and $\pi_{i+1} f \neq q$ and $p \notin \mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$. Given s. If $s > 0$, then there exists p_3 such that $p_3 \notin \mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$ and $p_3 \in \mathcal{L}(p,q)$ and $p_3 \in \operatorname{Ball}(u,s)$.

Let us consider f_1, f_2, P. We say that f_1 and f_2 are generators of P if and only if the conditions (Def.5) are satisfied.

(Def.5) (i) f_1 is alternating,
(ii) f_2 is alternating,
(iii) $\pi_1 f_1 = \pi_1 f_2$,
(iv) $\pi_{\operatorname{len} f_1} f_1 = \pi_{\operatorname{len} f_2} f_2$,
(v) $\langle \pi_2 f_1, \pi_1 f_1, \pi_2 f_2 \rangle$ is alternating,
(vi) $\langle \pi_{\operatorname{len} f_1 - 1} f_1, \pi_{\operatorname{len} f_1} f_1, \pi_{\operatorname{len} f_2 - 1} f_2 \rangle$ is alternating,
(vii) $\pi_1 f_1 \neq \pi_{\operatorname{len} f_1} f_1$,
(viii) $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) = \{\pi_1 f_1, \pi_{\operatorname{len} f_1} f_1\}$, and
(ix) $P = \widetilde{\mathcal{L}}(f_1) \cup \widetilde{\mathcal{L}}(f_2)$.

Next we state the proposition

(61) If f_1 and f_2 are generators of P and $1 < i$ and $i < \operatorname{len} f_1$, then $\pi_i f_1$ is extremal in P.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.

[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[8] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[10] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[11] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[13] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[14] Zbigniew Karno. Maximal discrete subspaces of almost discrete topological spaces. *Formalized Mathematics*, 4(**1**):125–135, 1993.
[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[16] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[17] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[19] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[20] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[22] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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Relocatability [1]

Yasushi Tanaka
Shinshu University
Information Engineering Dept.
Nagano

Summary. This article defines the concept of relocating the program part of a finite partial state of **SCM** (data part stays intact). The relocated program differs from the original program in that all jump instructions are adjusted by the relocation factor and other instructions remain unchanged. The main theorem states that if a program computes a function then the relocated program computes the same function, and vice versa.

MML Identifier: `RELOC`.

The terminology and notation used in this paper have been introduced in the following articles: [16], [2], [1], [19], [5], [6], [15], [7], [18], [13], [4], [9], [3], [8], [10], [11], [17], [12], and [14].

1. Relocatability

In this paper j, k, m will be natural numbers.

Let l_1 be an instruction-location of **SCM** and let k be a natural number. The functor l_1+k yielding an instruction-location of **SCM** is defined as follows:

(Def.1) There exists a natural number m such that $l_1 = \mathbf{i}_m$ and $l_1 + k = \mathbf{i}_{m+k}$.

The functor $l_1 -' k$ yields an instruction-location of **SCM** and is defined as follows:

(Def.2) There exists a natural number m such that $l_1 = \mathbf{i}_m$ and $l_1 -' k = \mathbf{i}_{m-'k}$.

The following three propositions are true:

(1) For every instruction-location l_1 of **SCM** and for every natural number k holds $(l_1 + k) -' k = l_1$.

[1]This work was done under guidance and supervision of A. Trybulec and P. Rudnicki.

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(2) For all instructions-locations l_2, l_3 of **SCM** and for every natural number k holds Start-At$(l_2 + k)$ = Start-At$(l_3 + k)$ iff Start-At(l_2) = Start-At(l_3).

(3) For all instructions-locations l_2, l_3 of **SCM** and for every natural number k such that Start-At(l_2) = Start-At(l_3) holds Start-At$(l_2 -' k)$ = Start-At$(l_3 -' k)$.

Let I be an instruction of **SCM** and let k be a natural number. The functor IncAddr(I, k) yields an instruction of **SCM** and is defined as follows:

(Def.3) (i) IncAddr(I, k) = goto $((^{@}I)\mathrm{address}_{\mathrm{j}}^{\mathrm{T}} + k)$ if InsCode$(I) = 6$,

(ii) IncAddr(I, k) = **if** $(^{@}I)\mathrm{address}_{\mathrm{c}}^{\mathrm{T}} = 0$ **goto** $(^{@}I)\mathrm{address}_{\mathrm{j}}^{\mathrm{T}} + k$ if InsCode$(I) = 7$,

(iii) IncAddr(I, k) = **if** $(^{@}I)\mathrm{address}_{\mathrm{c}}^{\mathrm{T}} > 0$ **goto** $(^{@}I)\mathrm{address}_{\mathrm{j}}^{\mathrm{T}} + k$ if InsCode$(I) = 8$,

(iv) IncAddr$(I, k) = I$, otherwise.

One can prove the following propositions:

(4) For every natural number k holds IncAddr$(\mathbf{halt}_{\mathbf{SCM}}, k) = \mathbf{halt}_{\mathbf{SCM}}$.

(5) For every natural number k and for all data-locations a, b holds IncAddr$(a{:=}b, k) = a{:=}b$.

(6) For every natural number k and for all data-locations a, b holds IncAddr(AddTo$(a, b), k)$ = AddTo(a, b).

(7) For every natural number k and for all data-locations a, b holds IncAddr(SubFrom$(a, b), k)$ = SubFrom(a, b).

(8) For every natural number k and for all data-locations a, b holds IncAddr(MultBy$(a, b), k)$ = MultBy(a, b).

(9) For every natural number k and for all data-locations a, b holds IncAddr(Divide$(a, b), k)$ = Divide(a, b).

(10) For every natural number k and for every instruction-location l_1 of **SCM** holds IncAddr(goto $l_1, k)$ = goto $(l_1 + k)$.

(11) Let k be a natural number, and let l_1 be an instruction-location of **SCM**, and let a be a data-location. Then IncAddr(**if** $a = 0$ **goto** $l_1, k)$ = **if** $a = 0$ **goto** $l_1 + k$.

(12) Let k be a natural number, and let l_1 be an instruction-location of **SCM**, and let a be a data-location. Then IncAddr(**if** $a > 0$ **goto** $l_1, k)$ = **if** $a > 0$ **goto** $l_1 + k$.

(13) For every instruction I of **SCM** and for every natural number k holds InsCode(IncAddr(I, k)) = InsCode(I).

(14) Let I_1, I be instructions of **SCM** and let k be a natural number. Suppose InsCode$(I) = 0$ or InsCode$(I) = 1$ or InsCode$(I) = 2$ or InsCode$(I) = 3$ or InsCode$(I) = 4$ or InsCode$(I) = 5$ but IncAddr$(I_1, k) = I$. Then $I_1 = I$.

Let p be a programmed finite partial state of **SCM** and let k be a natural number. The functor Shift(p, k) yielding a programmed finite partial state of

SCM is defined by:

(Def.4) dom Shift(p, k) = $\{\mathbf{i}_{m+k} : \mathbf{i}_m \in \operatorname{dom} p\}$ and for every m such that $\mathbf{i}_m \in \operatorname{dom} p$ holds $(\text{Shift}(p, k))(\mathbf{i}_{m+k}) = p(\mathbf{i}_m)$.

We now state three propositions:

(15) Let l be an instruction-location of **SCM**, and let k be a natural number, and let p be a programmed finite partial state of **SCM**. If $l \in \operatorname{dom} p$, then $(\text{Shift}(p, k))(l + k) = p(l)$.

(16) Let p be a programmed finite partial state of **SCM** and let k be a natural number. Then dom Shift(p, k) = $\{i_1 + k : i_1$ ranges over instructions-locations of **SCM**, $i_1 \in \operatorname{dom} p\}$.

(17) Let p be a programmed finite partial state of **SCM** and let k be a natural number. Then dom Shift$(p, k) \subseteq$ the instruction locations of **SCM**.

Let p be a programmed finite partial state of **SCM** and let k be a natural number. The functor IncAddr(p, k) yielding a programmed finite partial state of **SCM** is defined as follows:

(Def.5) dom IncAddr(p, k) = dom p and for every m such that $\mathbf{i}_m \in \operatorname{dom} p$ holds $(\text{IncAddr}(p, k))(\mathbf{i}_m) = \text{IncAddr}(\pi_{\mathbf{i}_m} p, k)$.

One can prove the following two propositions:

(18) Let p be a programmed finite partial state of **SCM**, and let k be a natural number, and let l be an instruction-location of **SCM**. If $l \in \operatorname{dom} p$, then $(\text{IncAddr}(p, k))(l) = \text{IncAddr}(\pi_l p, k)$.

(19) For every natural number i and for every programmed finite partial state p of **SCM** holds $\text{Shift}(\text{IncAddr}(p, i), i) = \text{IncAddr}(\text{Shift}(p, i), i)$.

Let p be a finite partial state of **SCM** and let k be a natural number. The functor Relocated(p, k) yielding a finite partial state of **SCM** is defined as follows:

(Def.6) $\text{Relocated}(p, k) = \text{Start-At}(\mathbf{IC}_p + k) +\cdot \text{IncAddr}(\text{Shift}(\text{ProgramPart}(p), k), k) +\cdot \text{DataPart}(p)$.

Next we state a number of propositions:

(20) For every finite partial state p of **SCM** holds dom IncAddr(Shift(ProgramPart$(p), k), k) \subseteq \text{Instr-Loc}_{\text{SCM}}$.

(21) For every finite partial state p of **SCM** and for every natural number k holds $\text{DataPart}(\text{Relocated}(p, k)) = \text{DataPart}(p)$.

(22) For every finite partial state p of **SCM** and for every natural number k holds $\text{ProgramPart}(\text{Relocated}(p, k)) = \text{IncAddr}(\text{Shift}(\text{ProgramPart}(p), k), k)$.

(23) For every finite partial state p of **SCM** holds dom ProgramPart(Relocated$(p, k)) = \{\mathbf{i}_{j+k} : \mathbf{i}_j \in \operatorname{dom} \text{ProgramPart}(p)\}$.

(24) Let p be a finite partial state of **SCM**, and let k be a natural number, and let l be an instruction-location of **SCM**. Then $l \in \operatorname{dom} p$ if and only if $l + k \in \operatorname{dom} \text{Relocated}(p, k)$.

(25) For every finite partial state p of **SCM** and for every natural number k holds $\mathbf{IC}_{\mathbf{SCM}} \in \operatorname{dom} \text{Relocated}(p, k)$.

(26) For every finite partial state p of $\mathbf{SCM}$ and for every natural number k holds $\mathbf{IC}_{\mathrm{Relocated}(p,k)} = \mathbf{IC}_p + k$.

(27) Let p be a finite partial state of $\mathbf{SCM}$, and let k be a natural number, and let l_1 be an instruction-location of $\mathbf{SCM}$, and let I be an instruction of $\mathbf{SCM}$. If $l_1 \in \mathrm{dom}\,\mathrm{ProgramPart}(p)$ and $I = p(l_1)$, then $\mathrm{IncAddr}(I, k) = (\mathrm{Relocated}(p, k))(l_1 + k)$.

(28) For every finite partial state p of $\mathbf{SCM}$ and for every natural number k holds $\text{Start-At}(\mathbf{IC}_p + k) \subseteq \mathrm{Relocated}(p, k)$.

(29) Let s be a data-only finite partial state of $\mathbf{SCM}$, and let p be a finite partial state of $\mathbf{SCM}$, and let k be a natural number. If $\mathbf{IC}_{\mathbf{SCM}} \in \mathrm{dom}\,p$, then $\mathrm{Relocated}(p +\cdot s, k) = \mathrm{Relocated}(p, k) +\cdot s$.

(30) Let k be a natural number, and let p be an autonomic finite partial state of $\mathbf{SCM}$, and let s_1, s_2 be states of $\mathbf{SCM}$. If $p \subseteq s_1$ and $\mathrm{Relocated}(p, k) \subseteq s_2$, then $p \subseteq s_1 +\cdot s_2 \restriction \text{Data-Loc}_{\mathrm{SCM}}$.

(31) For every state s of $\mathbf{SCM}$ holds $\mathrm{Exec}(\mathrm{IncAddr}(\mathrm{CurInstr}(s), k), s +\cdot \text{Start-At}(\mathbf{IC}_s + k)) = \mathrm{Following}(s) +\cdot \text{Start-At}(\mathbf{IC}_{\mathrm{Following}(s)} + k)$.

(32) Let I_2 be an instruction of $\mathbf{SCM}$, and let s be a state of $\mathbf{SCM}$, and let p be a finite partial state of $\mathbf{SCM}$, and let i, j, k be natural numbers. If $\mathbf{IC}_s = \mathbf{i}_{j+k}$, then $\mathrm{Exec}(I_2, s +\cdot \text{Start-At}(\mathbf{IC}_s -' k)) = \mathrm{Exec}(\mathrm{IncAddr}(I_2, k), s) +\cdot \text{Start-At}(\mathbf{IC}_{\mathrm{Exec}(\mathrm{IncAddr}(I_2,k),s)} -' k)$.

2. Main theorems of Relocatability

Next we state several propositions:

(33) Let k be a natural number and let p be an autonomic finite partial state of $\mathbf{SCM}$. Suppose $\mathbf{IC}_{\mathbf{SCM}} \in \mathrm{dom}\,p$. Let s be a state of $\mathbf{SCM}$. Suppose $p \subseteq s$. Let i be a natural number. Then $(\mathrm{Computation}(s +\cdot \mathrm{Relocated}(p, k)))(i) = (\mathrm{Computation}(s))(i) +\cdot \text{Start-At}(\mathbf{IC}_{(\mathrm{Computation}(s))(i)} + k) +\cdot \mathrm{ProgramPart}(\mathrm{Relocated}(p, k))$.

(34) Let k be a natural number, and let p be an autonomic finite partial state of $\mathbf{SCM}$, and let s_1, s_2, s_3 be states of $\mathbf{SCM}$. Suppose $\mathbf{IC}_{\mathbf{SCM}} \in \mathrm{dom}\,p$ and $p \subseteq s_1$ and $\mathrm{Relocated}(p, k) \subseteq s_2$ and $s_3 = s_1 +\cdot s_2 \restriction \text{Data-Loc}_{\mathrm{SCM}}$. Let i be a natural number. Then $\mathbf{IC}_{(\mathrm{Computation}(s_1))(i)} + k = \mathbf{IC}_{(\mathrm{Computation}(s_2))(i)}$ and $\mathrm{IncAddr}(\mathrm{CurInstr}((\mathrm{Computation}(s_1))(i)), k) = \mathrm{CurInstr}((\mathrm{Computation}(s_2))(i))$ and $(\mathrm{Computation}(s_1))(i) \restriction \mathrm{dom}\,\mathrm{DataPart}(p) = (\mathrm{Computation}(s_2))(i) \restriction \mathrm{dom}\,\mathrm{DataPart}(\mathrm{Relocated}(p, k))$ and $(\mathrm{Computation}(s_3))(i) \restriction \text{Data-Loc}_{\mathrm{SCM}} = (\mathrm{Computation}(s_2))(i) \restriction \text{Data-Loc}_{\mathrm{SCM}}$.

(35) Let p be an autonomic finite partial state of $\mathbf{SCM}$ and let k be a natural number. If $\mathbf{IC}_{\mathbf{SCM}} \in \mathrm{dom}\,p$, then p is halting iff $\mathrm{Relocated}(p, k)$ is halting.

(36) Let k be a natural number and let p be an autonomic finite partial state of $\mathbf{SCM}$. Suppose $\mathbf{IC}_{\mathbf{SCM}} \in \mathrm{dom}\,p$. Let s be a

state of **SCM**. Suppose $\text{Relocated}(p, k) \subseteq s$. Let i be a natural number. Then $(\text{Computation}(s))(i) = (\text{Computation}(s +\cdot p))(i) +\cdot \text{Start-At}(\mathbf{IC}_{(\text{Computation}(s+\cdot p))(i)} + k) +\cdot s \restriction \text{dom}\,\text{ProgramPart}(p) +\cdot \text{ProgramPart}(\text{Relocated}(p, k))$.

(37) Let k be a natural number and let p be a finite partial state of **SCM**. Suppose $\mathbf{IC}_{\mathbf{SCM}} \in \text{dom}\,p$. Let s be a state of **SCM**. Suppose $p \subseteq s$ and $\text{Relocated}(p, k)$ is autonomic. Let i be a natural number. Then $(\text{Computation}(s))(i) = (\text{Computation}(s +\cdot \text{Relocated}(p, k)))(i) +\cdot \text{Start-At}(\mathbf{IC}_{(\text{Computation}(s+\cdot \text{Relocated}(p,k)))(i)} -' k) +\cdot s \restriction \text{dom}\,\text{ProgramPart}(\text{Relocated}(p, k)) +\cdot \text{ProgramPart}(p)$.

(38) Let p be a finite partial state of **SCM**. Suppose $\mathbf{IC}_{\mathbf{SCM}} \in \text{dom}\,p$. Let k be a natural number. Then p is autonomic if and only if $\text{Relocated}(p, k)$ is autonomic.

(39) Let p be a halting autonomic finite partial state of **SCM**. If $\mathbf{IC}_{\mathbf{SCM}} \in \text{dom}\,p$, then for every natural number k holds $\text{DataPart}(\text{Result}(p)) = \text{DataPart}(\text{Result}(\text{Relocated}(p, k)))$.

(40) Let F be a data-only partial function from FinPartSt(**SCM**) to FinPartSt(**SCM**) and let p be a finite partial state of **SCM**. Suppose $\mathbf{IC}_{\mathbf{SCM}} \in \text{dom}\,p$. Let k be a natural number. Then p computes F if and only if $\text{Relocated}(p, k)$ computes F.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[8] Czesław Byliński. Products and coproducts in categories. *Formalized Mathematics*, 2(**5**):701–709, 1991.
[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[10] Yatsuka Nakamura and Andrzej Trybulec. A mathematical model of CPU. *Formalized Mathematics*, 3(**2**):151–160, 1992.
[11] Yatsuka Nakamura and Andrzej Trybulec. On a mathematical model of programs. *Formalized Mathematics*, 3(**2**):241–250, 1992.
[12] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.
[13] Dariusz Surowik. Cyclic groups and some of their properties - part I. *Formalized Mathematics*, 2(**5**):623–627, 1991.
[14] Yasushi Tanaka. On the decomposition of the states of SCM. *Formalized Mathematics*, 5(**1**):1–8, 1996.
[15] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[17] Andrzej Trybulec and Yatsuka Nakamura. Some remarks on the simple concrete model of computer. *Formalized Mathematics*, 4(**1**):51–56, 1993.

[18] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

[19] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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Maximal Anti-Discrete Subspaces of Topological Spaces

Zbigniew Karno
Warsaw University
Białystok

Summary. Let X be a topological space and let A be a subset of X. A is said to be *anti-discrete* provided for every open subset G of X either $A \cap G = \emptyset$ or $A \subseteq G$; equivalently, for every closed subset F of X either $A \cap F = \emptyset$ or $A \subseteq F$. An anti-discrete subset M of X is said to be *maximal anti-discrete* provided for every anti-discrete subset A of X if $M \subseteq A$ then $M = A$. A subspace of X is *maximal anti-discrete* iff its carrier is maximal anti-discrete in X. The purpose is to list a few properties of maximal anti-discrete sets and subspaces in Mizar formalism.

It is shown that every $x \in X$ is contained in a unique maximal anti-discrete subset $\mathrm{M}(x)$ of X, denoted in the text by MaxADSet(x). Such subset can be defined by

$$\mathrm{M}(x) = \bigcap \{S \subseteq X : x \in S, \text{ and } S \text{ is open or closed in } X\}.$$

It has the following remarkable properties: (1) $y \in \mathrm{M}(x)$ iff $\mathrm{M}(y) = \mathrm{M}(x)$, (2) either $\mathrm{M}(x) \cap \mathrm{M}(y) = \emptyset$ or $\mathrm{M}(x) = \mathrm{M}(y)$, (3) $\mathrm{M}(x) = \mathrm{M}(y)$ iff $\overline{\{x\}} = \overline{\{y\}}$, and (4) $\mathrm{M}(x) \cap \mathrm{M}(y) = \emptyset$ iff $\overline{\{x\}} \neq \overline{\{y\}}$. It follows from these properties that $\{\mathrm{M}(x) : x \in X\}$ is the T_0-partition of X defined by M.H. Stone in [7].

Moreover, it is shown that the operation M defined on all subsets of X by

$$\mathrm{M}(A) = \bigcup \{\mathrm{M}(x) : x \in A\},$$

denoted in the text by MaxADSet(A), satisfies the Kuratowski closure axioms (see e.g., [4]), i.e., (1) $\mathrm{M}(A \cup B) = \mathrm{M}(A) \cup \mathrm{M}(B)$, (2) $\mathrm{M}(A) = \mathrm{M}(\mathrm{M}(A))$, (3) $A \subseteq \mathrm{M}(A)$,and (4) $\mathrm{M}(\emptyset) = \emptyset$. Note that this operation commutes with the usual closure operation of X, and if A is an open (or a closed) subset of X, then $\mathrm{M}(A) = A$.

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The articles [11], [12], [8], [10], [5], [6], [13], [9], [3], [1], and [2] provide the terminology and notation for this paper.

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1. Properties of the Closure and the Interior Operations

Let X be a topological space and let A be a non empty subset of X. Observe that $\overline{A}$ is non empty.

Let X be a topological space and let A be an empty subset of X. One can check that $\overline{A}$ is empty.

Let X be a topological space and let A be a non proper subset of X. One can check that $\overline{A}$ is non proper.

Let X be a non trivial topological space and let A be a non trivial non empty subset of X. Observe that $\overline{A}$ is non trivial.

In the sequel X is a topological space.

We now state three propositions:

(1) For every subset A of X holds $\overline{A} = \bigcap\{F : F$ ranges over subsets of X, F is closed $\wedge$ $A \subseteq F\}$.

(2) For every point x of X holds $\overline{\{x\}} = \bigcap\{F : F$ ranges over subsets of X, F is closed $\wedge$ $x \in F\}$.

(3) For all subsets A, B of X such that $B \subseteq \overline{A}$ holds $\overline{B} \subseteq \overline{A}$.

Let X be a topological space and let A be a non proper subset of X. Note that $\operatorname{Int} A$ is non proper.

Let X be a topological space and let A be a proper subset of X. One can check that $\operatorname{Int} A$ is proper.

Let X be a topological space and let A be an empty subset of X. Note that $\operatorname{Int} A$ is empty.

Next we state two propositions:

(4) For every subset A of X holds $\operatorname{Int} A = \bigcup\{G : G$ ranges over subsets of X, G is open $\wedge$ $G \subseteq A\}$.

(5) For all subsets A, B of X such that $\operatorname{Int} A \subseteq B$ holds $\operatorname{Int} A \subseteq \operatorname{Int} B$.

2. Anti-Discrete Subsets of Topological Structures

Let Y be a topological structure. A subset of Y is anti-discrete if:

(Def.1) For every point x of Y and for every subset G of Y such that G is open and $x \in G$ holds if $x \in$ it, then it $\subseteq G$.

Let Y be a non empty topological structure. Let us observe that a subset of Y is anti-discrete if:

(Def.2) For every point x of Y and for every subset F of Y such that F is closed and $x \in F$ holds if $x \in$ it, then it $\subseteq F$.

Let Y be a topological structure. Let us observe that a subset of Y is anti-discrete if:

(Def.3) For every subset G of Y such that G is open holds it $\cap G = \emptyset$ or it $\subseteq G$.

Let Y be a topological structure. Let us observe that a subset of Y is anti-discrete if:

(Def.4) For every subset F of Y such that F is closed holds it $\cap F = \emptyset$ or it $\subseteq F$.

Next we state the proposition

(6) Let Y_0, Y_1 be topological structures, and let D_0 be a subset of Y_0, and let D_1 be a subset of Y_1. Suppose the topological structure of Y_0 = the topological structure of Y_1 and $D_0 = D_1$. If D_0 is anti-discrete, then D_1 is anti-discrete.

In the sequel Y will denote a non empty topological structure.

Next we state three propositions:

(7) For all subsets A, B of Y such that $B \subseteq A$ holds if A is anti-discrete, then B is anti-discrete.

(8) For every point x of Y holds $\{x\}$ is anti-discrete.

(9) Every empty subset of Y is anti-discrete.

Let Y be a topological structure. A family of subsets of Y is anti-discrete-set-family if:

(Def.5) For every subset A of Y such that $A \in$ it holds A is anti-discrete.

One can prove the following propositions:

(10) Let F be a family of subsets of Y. Suppose F is anti-discrete-set-family. If $\bigcap F \neq \emptyset$, then $\bigcup F$ is anti-discrete.

(11) For every family F of subsets of Y such that F is anti-discrete-set-family holds $\bigcap F$ is anti-discrete.

Let Y be a non empty topological structure and let x be a point of Y. The functor MaxADSF(x) yields a non empty family of subsets of Y and is defined by:

(Def.6) $\text{MaxADSF}(x) = \{A : A$ ranges over subsets of Y, A is anti-discrete $\wedge\ x \in A\}$.

In the sequel x will denote a point of Y.

We now state four propositions:

(12) MaxADSF(x) is anti-discrete-set-family.

(13) $\{x\} = \bigcap \text{MaxADSF}(x)$.

(14) $\{x\} \subseteq \bigcup \text{MaxADSF}(x)$.

(15) $\bigcup \text{MaxADSF}(x)$ is anti-discrete.

3. Maximal Anti-Discrete Subsets of Topological Structures

Let Y be a topological structure. A subset of Y is maximal anti-discrete if:

(Def.7) It is anti-discrete and for every subset D of Y such that D is anti-discrete and it $\subseteq D$ holds it $= D$.

We now state the proposition

(16) Let Y_0, Y_1 be topological structures, and let D_0 be a subset of Y_0, and let D_1 be a subset of Y_1. Suppose the topological structure of $Y_0 =$ the topological structure of Y_1 and $D_0 = D_1$. If D_0 is maximal anti-discrete, then D_1 is maximal anti-discrete.

In the sequel Y will denote a non empty topological structure.

One can prove the following propositions:

(17) Every empty subset of Y is not maximal anti-discrete.

(18) For every non empty subset A of Y such that A is anti-discrete and open holds A is maximal anti-discrete.

(19) For every non empty subset A of Y such that A is anti-discrete and closed holds A is maximal anti-discrete.

Let Y be a non empty topological structure and let x be a point of Y. The functor $\mathrm{MaxADSet}(x)$ yielding a non empty subset of Y is defined by:

(Def.8) $\mathrm{MaxADSet}(x) = \bigcup \mathrm{MaxADSF}(x)$.

We now state several propositions:

(20) For every point x of Y holds $\{x\} \subseteq \mathrm{MaxADSet}(x)$.

(21) For every subset D of Y and for every point x of Y such that D is anti-discrete and $x \in D$ holds $D \subseteq \mathrm{MaxADSet}(x)$.

(22) For every point x of Y holds $\mathrm{MaxADSet}(x)$ is maximal anti-discrete.

(23) For all points x, y of Y holds $y \in \mathrm{MaxADSet}(x)$ iff $\mathrm{MaxADSet}(y) = \mathrm{MaxADSet}(x)$.

(24) For all points x, y of Y holds $\mathrm{MaxADSet}(x) \cap \mathrm{MaxADSet}(y) = \emptyset$ or $\mathrm{MaxADSet}(x) = \mathrm{MaxADSet}(y)$.

(25) For every subset F of Y and for every point x of Y such that F is closed and $x \in F$ holds $\mathrm{MaxADSet}(x) \subseteq F$.

(26) For every subset G of Y and for every point x of Y such that G is open and $x \in G$ holds $\mathrm{MaxADSet}(x) \subseteq G$.

(27) Let x be a point of Y. Suppose $\{F : F$ ranges over subsets of Y, F is closed $\wedge\ x \in F\} \neq \emptyset$. Then $\mathrm{MaxADSet}(x) \subseteq \bigcap\{F : F$ ranges over subsets of Y, F is closed $\wedge\ x \in F\}$.

(28) Let x be a point of Y. Suppose $\{G : G$ ranges over subsets of Y, G is open $\wedge\ x \in G\} \neq \emptyset$. Then $\mathrm{MaxADSet}(x) \subseteq \bigcap\{G : G$ ranges over subsets of Y, G is open $\wedge\ x \in G\}$.

Let Y be a non empty topological structure. Let us observe that a subset of Y is maximal anti-discrete if:

(Def.9) There exists a point x of Y such that $x \in$ it and it $= \mathrm{MaxADSet}(x)$.

The following proposition is true

(29) For every subset A of Y and for every point x of Y such that $x \in A$ holds if A is maximal anti-discrete, then $A = \mathrm{MaxADSet}(x)$.

Let Y be a non empty topological structure. Let us observe that a non empty subset of Y is maximal anti-discrete if:

(Def.10) For every point x of Y such that $x \in$ it holds it $=$ MaxADSet(x).

Let Y be a non empty topological structure and let A be a subset of Y. The functor MaxADSet(A) yielding a subset of Y is defined as follows:

(Def.11) MaxADSet$(A) = \bigcup\{$MaxADSet$(a) : a$ ranges over points of Y, $a \in A\}$.

Next we state a number of propositions:

(30) For every point x of Y holds MaxADSet$(x) =$ MaxADSet$(\{x\})$.

(31) For every subset A of Y and for every point x of Y such that MaxADSet$(x) \cap$ MaxADSet$(A) \neq \emptyset$ holds MaxADSet$(x) \cap A \neq \emptyset$.

(32) For every subset A of Y and for every point x of Y such that MaxADSet$(x) \cap$ MaxADSet$(A) \neq \emptyset$ holds MaxADSet$(x) \subseteq$ MaxADSet(A).

(33) For all subsets A, B of Y such that $A \subseteq B$ holds MaxADSet$(A) \subseteq$ MaxADSet(B).

(34) For every subset A of Y holds $A \subseteq$ MaxADSet(A).

(35) For every subset A of Y holds MaxADSet$(A) =$ MaxADSet(MaxADSet(A)).

(36) For all subsets A, B of Y such that $A \subseteq$ MaxADSet(B) holds MaxADSet$(A) \subseteq$ MaxADSet(B).

(37) For all subsets A, B of Y holds $B \subseteq$ MaxADSet(A) and $A \subseteq$ MaxADSet(B) iff MaxADSet$(A) =$ MaxADSet(B).

(38) For all subsets A, B of Y holds MaxADSet$(A \cup B) =$ MaxADSet$(A) \cup$ MaxADSet(B).

(39) For all subsets A, B of Y holds MaxADSet$(A \cap B) \subseteq$ MaxADSet$(A) \cap$ MaxADSet(B).

Let Y be a non empty topological structure and let A be a non empty subset of Y. One can verify that MaxADSet(A) is non empty.

Let Y be a non empty topological structure and let A be an empty subset of Y. One can verify that MaxADSet(A) is empty.

Let Y be a non empty topological structure and let A be a non proper subset of Y. Observe that MaxADSet(A) is non proper.

Let Y be a non trivial non empty topological structure and let A be a non trivial non empty subset of Y. Note that MaxADSet(A) is non trivial.

The following four propositions are true:

(40) For every subset G of Y and for every subset A of Y such that G is open and $A \subseteq G$ holds MaxADSet$(A) \subseteq G$.

(41) Let A be a subset of Y. Suppose $\{G : G$ ranges over subsets of Y, G is open $\wedge\ A \subseteq G\} \neq \emptyset$. Then MaxADSet$(A) \subseteq \bigcap\{G : G$ ranges over subsets of Y, G is open $\wedge\ A \subseteq G\}$.

(42) For every subset F of Y and for every subset A of Y such that F is closed and $A \subseteq F$ holds MaxADSet$(A) \subseteq F$.

(43) Let A be a subset of Y. Suppose $\{F : F$ ranges over subsets of Y, F is closed $\wedge$ $A \subseteq F\} \neq \emptyset$. Then $\mathrm{MaxADSet}(A) \subseteq \bigcap\{F : F$ ranges over subsets of Y, F is closed $\wedge$ $A \subseteq F\}$.

4. Anti-Discrete and Maximal Anti-discrete Subsets of Topological Spaces

Let X be a topological space. Let us observe that a subset of X is anti-discrete if:

(Def.12) For every point x of X such that $x \in$ it holds it $\subseteq \overline{\{x\}}$.

Let X be a topological space. Let us observe that a subset of X is anti-discrete if:

(Def.13) For every point x of X such that $x \in$ it holds $\overline{\text{it}} = \overline{\{x\}}$.

Let X be a topological space. Let us observe that a subset of X is anti-discrete if:

(Def.14) For all points x, y of X such that $x \in$ it and $y \in$ it holds $\overline{\{x\}} = \overline{\{y\}}$.

In the sequel X will be a topological space.

The following four propositions are true:

(44) For every point x of X and for every subset D of X such that D is anti-discrete and $\overline{\{x\}} \subseteq D$ holds $D = \overline{\{x\}}$.

(45) Let A be a subset of X. Then A is anti-discrete and closed if and only if for every point x of X such that $x \in A$ holds $A = \overline{\{x\}}$.

(46) For every subset A of X such that A is anti-discrete and A is not open holds A is boundary.

(47) For every point x of X such that $\overline{\{x\}} = \{x\}$ holds $\{x\}$ is maximal anti-discrete.

In the sequel x, y will be points of X.

The following propositions are true:

(48) $\mathrm{MaxADSet}(x) \subseteq \bigcap\{G : G$ ranges over subsets of X, G is open $\wedge$ $x \in G\}$.

(49) $\mathrm{MaxADSet}(x) \subseteq \bigcap\{F : F$ ranges over subsets of X, F is closed $\wedge$ $x \in F\}$.

(50) $\mathrm{MaxADSet}(x) \subseteq \overline{\{x\}}$.

(51) $\mathrm{MaxADSet}(x) = \mathrm{MaxADSet}(y)$ iff $\overline{\{x\}} = \overline{\{y\}}$.

(52) $\mathrm{MaxADSet}(x) \cap \mathrm{MaxADSet}(y) = \emptyset$ iff $\overline{\{x\}} \neq \overline{\{y\}}$.

Let X be a topological space and let x be a point of X. Then $\mathrm{MaxADSet}(x)$ is a non empty subset of X and it can be characterized by the condition:

(Def.15) $\mathrm{MaxADSet}(x) = \overline{\{x\}} \cap \bigcap\{G : G$ ranges over subsets of X, G is open $\wedge$ $x \in G\}$.

The following propositions are true:

(53) Let x, y be points of X. Then $\overline{\{x\}} \subseteq \overline{\{y\}}$ if and only if $\bigcap\{G : G$ ranges over subsets of X, G is open $\wedge\ y \in G\} \subseteq \bigcap\{G : G$ ranges over subsets of X, G is open $\wedge\ x \in G\}$.

(54) For all points x, y of X holds $\overline{\{x\}} \subseteq \overline{\{y\}}$ iff $\mathrm{MaxADSet}(y) \subseteq \bigcap\{G : G$ ranges over subsets of X, G is open $\wedge\ x \in G\}$.

(55) Let x, y be points of X. Then $\mathrm{MaxADSet}(x) \cap \mathrm{MaxADSet}(y) = \emptyset$ if and only if one of the following conditions is satisfied:

(i) there exists a subset V of X such that V is open and $\mathrm{MaxADSet}(x) \subseteq V$ and $V \cap \mathrm{MaxADSet}(y) = \emptyset$, or

(ii) there exists a subset W of X such that W is open and $W \cap \mathrm{MaxADSet}(x) = \emptyset$ and $\mathrm{MaxADSet}(y) \subseteq W$.

(56) Let x, y be points of X. Then $\mathrm{MaxADSet}(x) \cap \mathrm{MaxADSet}(y) = \emptyset$ if and only if one of the following conditions is satisfied:

(i) there exists a subset E of X such that E is closed and $\mathrm{MaxADSet}(x) \subseteq E$ and $E \cap \mathrm{MaxADSet}(y) = \emptyset$, or

(ii) there exists a subset F of X such that F is closed and $F \cap \mathrm{MaxADSet}(x) = \emptyset$ and $\mathrm{MaxADSet}(y) \subseteq F$.

In the sequel A, B denote subsets of X.

The following propositions are true:

(57) $\mathrm{MaxADSet}(A) \subseteq \bigcap\{G : G$ ranges over subsets of X, G is open $\wedge\ A \subseteq G\}$.

(58) If A is open, then $\mathrm{MaxADSet}(A) = A$.

(59) $\mathrm{MaxADSet}(\mathrm{Int}\, A) = \mathrm{Int}\, A$.

(60) $\mathrm{MaxADSet}(A) \subseteq \bigcap\{F : F$ ranges over subsets of X, F is closed $\wedge\ A \subseteq F\}$.

(61) $\mathrm{MaxADSet}(A) \subseteq \overline{A}$.

(62) If A is closed, then $\mathrm{MaxADSet}(A) = A$.

(63) $\mathrm{MaxADSet}(\overline{A}) = \overline{A}$.

(64) $\overline{\mathrm{MaxADSet}(A)} = \overline{A}$.

(65) If $\mathrm{MaxADSet}(A) = \mathrm{MaxADSet}(B)$, then $\overline{A} = \overline{B}$.

(66) If A is closed or B is closed, then $\mathrm{MaxADSet}(A \cap B) = \mathrm{MaxADSet}(A) \cap \mathrm{MaxADSet}(B)$.

(67) If A is open or B is open, then $\mathrm{MaxADSet}(A \cap B) = \mathrm{MaxADSet}(A) \cap \mathrm{MaxADSet}(B)$.

5. Maximal Anti-Discrete Subspaces

In the sequel Y is a non empty topological structure.

One can prove the following two propositions:

(68) Let Y_0 be a subspace of Y and let A be a subset of Y. Suppose $A =$ the carrier of Y_0. If Y_0 is anti-discrete, then A is anti-discrete.

(69) Let Y_0 be a subspace of Y. Suppose Y_0 is topological space-like. Let A be a subset of Y. Suppose $A =$ the carrier of Y_0. If A is anti-discrete, then Y_0 is anti-discrete.

In the sequel X will be a topological space and Y_0 will be a subspace of X.

One can prove the following four propositions:

(70) If for every open subspace X_0 of X holds Y_0 misses X_0 or Y_0 is a subspace of X_0, then Y_0 is anti-discrete.

(71) If for every closed subspace X_0 of X holds Y_0 misses X_0 or Y_0 is a subspace of X_0, then Y_0 is anti-discrete.

(72) Let Y_0 be an anti-discrete subspace of X and let X_0 be an open subspace of X. Then Y_0 misses X_0 or Y_0 is a subspace of X_0.

(73) Let Y_0 be an anti-discrete subspace of X and let X_0 be a closed subspace of X. Then Y_0 misses X_0 or Y_0 is a subspace of X_0.

Let Y be a non empty topological structure. A subspace of Y is maximal anti-discrete if it satisfies the conditions (Def.16).

(Def.16) (i) It is anti-discrete, and

(ii) for every subspace Y_0 of Y such that Y_0 is anti-discrete holds if the carrier of it $\subseteq$ the carrier of Y_0, then the carrier of it $=$ the carrier of Y_0.

Let Y be a non empty topological structure. Note that every subspace of Y which is maximal anti-discrete is also anti-discrete and every subspace of Y which is non anti-discrete is also non maximal anti-discrete.

Next we state the proposition

(74) Let Y_0 be a subspace of X and let A be a subset of X. Suppose $A =$ the carrier of Y_0. Then Y_0 is maximal anti-discrete if and only if A is maximal anti-discrete.

Let X be a topological space. One can check the following observations:

* every subspace of X which is open and anti-discrete is also maximal anti-discrete,
* every subspace of X which is open and non maximal anti-discrete is also non anti-discrete,
* every subspace of X which is anti-discrete and non maximal anti-discrete is also non open,
* every subspace of X which is closed and anti-discrete is also maximal anti-discrete,
* every subspace of X which is closed and non maximal anti-discrete is also non anti-discrete, and
* every subspace of X which is anti-discrete and non maximal anti-discrete is also non closed.

Let Y be a non empty topological structure and let x be a point of Y. The functor MaxADSspace(x) yielding a strict subspace of Y is defined by:

(Def.17) The carrier of MaxADSspace(x) $=$ MaxADSet(x).

We now state three propositions:

(75) For every point x of Y holds $\mathrm{Sspace}(x)$ is a subspace of $\mathrm{MaxADSspace}(x)$.

(76) Let x, y be points of Y. Then y is a point of $\mathrm{MaxADSspace}(x)$ if and only if the topological structure of $\mathrm{MaxADSspace}(y)$ = the topological structure of $\mathrm{MaxADSspace}(x)$.

(77) Let x, y be points of Y. Then

(i) the carrier of $\mathrm{MaxADSspace}(x)$ misses the carrier of $\mathrm{MaxADSspace}(y)$, or

(ii) the topological structure of $\mathrm{MaxADSspace}(x)$ = the topological structure of $\mathrm{MaxADSspace}(y)$.

Let X be a topological space. One can check that there exists a subspace of X which is maximal anti-discrete and strict.

Let X be a topological space and let x be a point of X. One can check that $\mathrm{MaxADSspace}(x)$ is maximal anti-discrete.

One can prove the following propositions:

(78) Let X_0 be a closed subspace of X and let x be a point of X. If x is a point of X_0, then $\mathrm{MaxADSspace}(x)$ is a subspace of X_0.

(79) Let X_0 be an open subspace of X and let x be a point of X. If x is a point of X_0, then $\mathrm{MaxADSspace}(x)$ is a subspace of X_0.

(80) For every point x of X such that $\overline{\{x\}} = \{x\}$ holds $\mathrm{Sspace}(x)$ is maximal anti-discrete.

Let Y be a non empty topological structure and let A be a non empty subset of Y. The functor $\mathrm{Sspace}(A)$ yielding a strict subspace of Y is defined by:

(Def.18) The carrier of $\mathrm{Sspace}(A) = A$.

One can prove the following propositions:

(81) Every non empty subset of Y is a subset of $\mathrm{Sspace}(A)$.

(82) Let Y_0 be a subspace of Y and let A be a non empty subset of Y. If A is a subset of Y_0, then $\mathrm{Sspace}(A)$ is a subspace of Y_0.

Let Y be a non trivial non empty topological structure. Note that there exists a subspace of Y which is non proper and strict.

Let Y be a non trivial non empty topological structure and let A be a non trivial non empty subset of Y. Observe that $\mathrm{Sspace}(A)$ is non trivial.

Let Y be a non empty topological structure and let A be a non proper non empty subset of Y. One can verify that $\mathrm{Sspace}(A)$ is non proper.

Let Y be a non empty topological structure and let A be a non empty subset of Y. The functor $\mathrm{MaxADSspace}(A)$ yields a strict subspace of Y and is defined by:

(Def.19) The carrier of $\mathrm{MaxADSspace}(A) = \mathrm{MaxADSet}(A)$.

We now state several propositions:

(83) Every non empty subset of Y is a subset of $\mathrm{MaxADSspace}(A)$.

(84) For every non empty subset A of Y holds $\mathrm{Sspace}(A)$ is a subspace of $\mathrm{MaxADSspace}(A)$.

(85) For every point x of Y holds the topological structure of MaxADSspace(x) = the topological structure of MaxADSspace($\{x\}$).

(86) For all non empty subsets A, B of Y such that $A \subseteq B$ holds MaxADSspace(A) is a subspace of MaxADSspace(B).

(87) For every non empty subset A of Y holds the topological structure of MaxADSspace(A) = the topological structure of MaxADSspace(MaxADSet(A)).

(88) For all non empty subsets A, B of Y such that A is a subset of MaxADSspace(B) holds MaxADSspace(A) is a subspace of MaxADSspace(B).

(89) Let A, B be non empty subsets of Y. Then B is a subset of MaxADSspace(A) and A is a subset of MaxADSspace(B) if and only if the topological structure of MaxADSspace(A) = the topological structure of MaxADSspace(B).

Let Y be a non trivial non empty topological structure and let A be a non trivial non empty subset of Y. One can verify that MaxADSspace(A) is non trivial.

Let Y be a non empty topological structure and let A be a non proper non empty subset of Y. One can verify that MaxADSspace(A) is non proper.

The following two propositions are true:

(90) Let X_0 be an open subspace of X and let A be a non empty subset of X. If A is a subset of X_0, then MaxADSspace(A) is a subspace of X_0.

(91) Let X_0 be a closed subspace of X and let A be a non empty subset of X. If A is a subset of X_0, then MaxADSspace(A) is a subspace of X_0.

References

[1] Zbigniew Karno. The lattice of domains of an extremally disconnected space. *Formalized Mathematics*, 3(**2**):143–149, 1992.

[2] Zbigniew Karno. Maximal discrete subspaces of almost discrete topological spaces. *Formalized Mathematics*, 4(**1**):125–135, 1993.

[3] Zbigniew Karno. Separated and weakly separated subspaces of topological spaces. *Formalized Mathematics*, 2(**5**):665–674, 1991.

[4] Kazimierz Kuratowski. *Topology.* Volume I, PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.

[5] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[6] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[7] M. H. Stone. Application of boolean algebras to topology. *Math. Sb.*, 1:765–771, 1936.

[8] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[9] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.

[10] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[12] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[13] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

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On Kolmogorov Topological Spaces [1]

Zbigniew Karno
Warsaw University
Białystok

Summary. Let X be a topological space. X is said to be T_0-*space* (or *Kolmogorov space*) provided for every pair of distinct points $x, y \in X$ there exists an open subset of X containing exactly one of these points; equivalently, for every pair of distinct points $x, y \in X$ there exists a closed subset of X containing exactly one of these points (see [1], [6], [2]).

The purpose is to list some of the standard facts on Kolmogorov spaces, using Mizar formalism. As a sample we formulate the following characteristics of such spaces: *X is a Kolmogorov space iff for every pair of distinct points $x, y \in X$ the closures $\overline{\{x\}}$ and $\overline{\{y\}}$ are distinct.*

There is also reviewed analogous facts on Kolmogorov subspaces of topological spaces. In the presented approach T_0-subsets are introduced and some of their properties developed.

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The articles [10], [11], [9], [7], [8], [5], [4], and [3] provide the terminology and notation for this paper.

1. Subspaces

Let Y be a non empty topological structure. We see that the subspace of Y is a non empty topological structure and it can be characterized by the following (equivalent) condition:

(Def.1) (i) The carrier of it $\subseteq$ the carrier of Y, and

(ii) for every subset G_0 of it holds G_0 is open iff there exists a subset G of Y such that G is open and $G_0 = G \cap$ (the carrier of it).

Next we state two propositions:

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(1) Let Y be a non empty topological structure, and let Y_0 be a subspace of Y, and let G_0 be a subset of Y_0. Then G_0 is open if and only if there exists a subset G of Y such that G is open and $G_0 = G \cap$ (the carrier of Y_0).

(2) Let Y be a non empty topological structure, and let Y_0 be a subspace of Y, and let G be a subset of Y. Suppose G is open. Then there exists a subset G_0 of Y_0 such that G_0 is open and $G_0 = G \cap$ (the carrier of Y_0).

Let Y be a non empty topological structure. We see that the subspace of Y is a non empty topological structure and it can be characterized by the following (equivalent) condition:

(Def.2) (i) The carrier of it $\subseteq$ the carrier of Y, and

(ii) for every subset F_0 of it holds F_0 is closed iff there exists a subset F of Y such that F is closed and $F_0 = F \cap$ (the carrier of it).

We now state two propositions:

(3) Let Y be a non empty topological structure, and let Y_0 be a subspace of Y, and let F_0 be a subset of Y_0. Then F_0 is closed if and only if there exists a subset F of Y such that F is closed and $F_0 = F \cap$ (the carrier of Y_0).

(4) Let Y be a non empty topological structure, and let Y_0 be a subspace of Y, and let F be a subset of Y. Suppose F is closed. Then there exists a subset F_0 of Y_0 such that F_0 is closed and $F_0 = F \cap$ (the carrier of Y_0).

2. Kolmogorov Spaces

A topological structure is T_0 if it satisfies the condition (Def.3).

(Def.3) Let x, y be points of it. Suppose $x \neq y$. Then

(i) there exists a subset V of it such that V is open and $x \in V$ and $y \notin V$, or

(ii) there exists a subset W of it such that W is open and $x \notin W$ and $y \in W$.

Let us observe that a non empty topological structure is T_0 if it satisfies the condition (Def.4).

(Def.4) Let x, y be points of it. Suppose $x \neq y$. Then

(i) there exists a subset E of it such that E is closed and $x \in E$ and $y \notin E$, or

(ii) there exists a subset F of it such that F is closed and $x \notin F$ and $y \in F$.

Let us mention that every non empty topological structure which is trivial is also T_0 and every non empty topological structure which is non T_0 is also non trivial.

One can verify that there exists a topological space which is strict T_0 and non empty and there exists a topological space which is strict non T_0 and non empty.

One can check the following observations:

* every topological space which is discrete is also T_0,
* every topological space which is non T_0 is also non discrete,
* every topological space which is anti-discrete and non trivial is also non T_0,
* every topological space which is anti-discrete and T_0 is also trivial, and
* every topological space which is T_0 and non trivial is also non anti-discrete.

Let us observe that a topological space is T_0 if:

(Def.5) For all points x, y of it such that $x \neq y$ holds $\overline{\{x\}} \neq \overline{\{y\}}$.

Let us observe that a topological space is T_0 if:

(Def.6) For all points x, y of it such that $x \neq y$ holds $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$.

Let us observe that a topological space is T_0 if:

(Def.7) For all points x, y of it such that $x \neq y$ and $x \in \overline{\{y\}}$ holds $\overline{\{y\}} \not\subseteq \overline{\{x\}}$.

One can verify the following observations:

* every topological space which is almost discrete and T_0 is also discrete,
* every topological space which is almost discrete and non discrete is also non T_0, and
* every topological space which is non discrete and T_0 is also non almost discrete.

A Kolmogorov space is a T_0 topological space. A non-Kolmogorov space is a non T_0 topological space.

Let us observe that there exists a Kolmogorov space which is non trivial and strict and there exists a non-Kolmogorov space which is non trivial and strict.

3. T_0-Subsets

Let Y be a topological structure. A subset of Y is T_0 if it satisfies the condition (Def.8).

(Def.8) Let x, y be points of Y. Suppose $x \in$ it and $y \in$ it and $x \neq y$. Then there exists a subset V of Y such that V is open and $x \in V$ and $y \notin V$ or there exists a subset W of Y such that W is open and $x \notin W$ and $y \in W$.

Let Y be a non empty topological structure. Let us observe that a subset of Y is T_0 if it satisfies the condition (Def.9).

(Def.9) Let x, y be points of Y. Suppose $x \in$ it and $y \in$ it and $x \neq y$. Then

(i) there exists a subset E of Y such that E is closed and $x \in E$ and $y \notin E$, or

(ii) there exists a subset F of Y such that F is closed and $x \notin F$ and $y \in F$.

Next we state two propositions:

(5) Let Y_0, Y_1 be topological structures, and let D_0 be a subset of Y_0, and let D_1 be a subset of Y_1. Suppose the topological structure of Y_0 = the topological structure of Y_1 and $D_0 = D_1$. If D_0 is T_0, then D_1 is T_0.

(6) Let Y be a non empty topological structure and let A be a subset of Y. Suppose A = the carrier of Y. Then A is T_0 if and only if Y is T_0.

In the sequel Y will denote a non empty topological structure.

The following propositions are true:

(7) For all subsets A, B of Y such that $B \subseteq A$ holds if A is T_0, then B is T_0.

(8) For all subsets A, B of Y such that A is T_0 or B is T_0 holds $A \cap B$ is T_0.

(9) Let A, B be subsets of Y. Suppose A is open or B is open. If A is T_0 and B is T_0, then $A \cup B$ is T_0.

(10) Let A, B be subsets of Y. Suppose A is closed or B is closed. If A is T_0 and B is T_0, then $A \cup B$ is T_0.

(11) For every subset A of Y such that A is discrete holds A is T_0.

(12) For every non empty subset A of Y such that A is anti-discrete and A is not trivial holds A is not T_0.

Let X be a topological space. Let us observe that a subset of X is T_0 if:

(Def.10) For all points x, y of X such that $x \in$ it and $y \in$ it and $x \neq y$ holds $\overline{\{x\}} \neq \overline{\{y\}}$.

Let X be a topological space. Let us observe that a subset of X is T_0 if:

(Def.11) For all points x, y of X such that $x \in$ it and $y \in$ it and $x \neq y$ holds $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$.

Let X be a topological space. Let us observe that a subset of X is T_0 if:

(Def.12) For all points x, y of X such that $x \in$ it and $y \in$ it and $x \neq y$ holds if $x \in \overline{\{y\}}$, then $\overline{\{y\}} \not\subseteq \overline{\{x\}}$.

In the sequel X will denote a topological space.

The following two propositions are true:

(13) Every empty subset of X is T_0.

(14) For every point x of X holds $\{x\}$ is T_0.

4. Kolmogorov Subspaces

Let Y be a non empty topological structure. Observe that there exists a subspace of Y which is strict and T_0.

Let Y be a non empty topological structure. Let us observe that a subspace of Y is T_0 if it satisfies the condition (Def.13).

(Def.13) Let x, y be points of Y. Suppose x is a point of it and y is a point of it and $x \neq y$. Then there exists a subset V of Y such that V is open and

$x \in V$ and $y \notin V$ or there exists a subset W of Y such that W is open and $x \notin W$ and $y \in W$.

Let Y be a non empty topological structure. Let us observe that a subspace of Y is T_0 if it satisfies the condition (Def.14).

(Def.14) Let x, y be points of Y. Suppose x is a point of it and y is a point of it and $x \neq y$. Then

(i) there exists a subset E of Y such that E is closed and $x \in E$ and $y \notin E$, or

(ii) there exists a subset F of Y such that F is closed and $x \notin F$ and $y \in F$.

In the sequel Y denotes a non empty topological structure.

The following propositions are true:

(15) Let Y_0 be a subspace of Y and let A be a subset of Y. Suppose $A =$ the carrier of Y_0. Then A is T_0 if and only if Y_0 is T_0.

(16) Let Y_0 be a subspace of Y and let Y_1 be a T_0 subspace of Y. If Y_0 is a subspace of Y_1, then Y_0 is T_0.

Let X be a topological space. One can check that there exists a subspace of X which is strict and T_0.

In the sequel X is a topological space.

The following propositions are true:

(17) For every T_0 subspace X_1 of X and for every subspace X_2 of X such that X_1 meets X_2 holds $X_1 \cap X_2$ is T_0.

(18) For all T_0 subspaces X_1, X_2 of X such that X_1 is open or X_2 is open holds $X_1 \cup X_2$ is T_0.

(19) For all T_0 subspaces X_1, X_2 of X such that X_1 is closed or X_2 is closed holds $X_1 \cup X_2$ is T_0.

Let X be a topological space. A Kolmogorov subspace of X is a T_0 subspace of X.

Next we state the proposition

(20) Let X be a topological space and let A_0 be a non empty subset of X. Suppose A_0 is T_0. Then there exists a strict Kolmogorov subspace X_0 of X such that $A_0 =$ the carrier of X_0.

Let X be a non trivial topological space. One can verify that there exists a Kolmogorov subspace of X which is proper and strict.

Let X be a Kolmogorov space. Observe that every subspace of X is T_0.

Let X be a non-Kolmogorov space. One can check that every subspace of X which is non proper is also non T_0 and every subspace of X which is T_0 is also proper.

Let X be a non-Kolmogorov space. Note that there exists a subspace of X which is strict and non T_0.

Let X be a non-Kolmogorov space. A non-Kolmogorov subspace of X is a non T_0 subspace of X.

We now state the proposition

(21) Let X be a non-Kolmogorov space and let A_0 be a subset of X. Suppose A_0 is not T_0. Then there exists a strict non-Kolmogorov subspace X_0 of X such that $A_0 =$ the carrier of X_0.

References

[1] P. Alexandroff and H. H. Hopf. *Topologie I.* Springer-Verlag, Berlin, 1935.
[2] Ryszard Engelking. *General Topology.* Volume 60 of *Monografie Matematyczne*, PWN - Polish Scientific Publishers, Warsaw, 1977.
[3] Zbigniew Karno. Maximal anti-discrete subspaces of topological spaces. *Formalized Mathematics*, 5(**1**):109–118, 1996.
[4] Zbigniew Karno. Maximal discrete subspaces of almost discrete topological spaces. *Formalized Mathematics*, 4(**1**):125–135, 1993.
[5] Zbigniew Karno. Separated and weakly separated subspaces of topological spaces. *Formalized Mathematics*, 2(**5**):665–674, 1991.
[6] Kazimierz Kuratowski. *Topology.* Volume I, PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.
[7] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[8] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[9] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[11] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

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Maximal Kolmogorov Subspaces of a Topological Space as Stone Retracts of the Ambient Space [1]

Zbigniew Karno
Warsaw University
Białystok

Summary. Let X be a topological space. X is said to be T_0*-space* (or *Kolmogorov space*) provided for every pair of distinct points x, $y \in X$ there exists an open subset of X containing exactly one of these points (see [1], [8], [4]). Such spaces and subspaces were investigated in Mizar formalism in [7]. A Kolmogorov subspace X_0 of a topological space X is said to be *maximal* provided for every Kolmogorov subspace Y of X if X_0 is subspace of Y then the topological structures of Y and X_0 are the same.

M.H. Stone proved in [10] that every topological space can be made into a Kolmogorov space by identifying points with the same closure (see also [11]). The purpose is to generalize the Stone result, using Mizar System. It is shown here that: (1) *in every topological space X there exists a maximal Kolmogorov subspace X_0 of X*, and (2) *every maximal Kolmogorov subspace X_0 of X is a continuous retract of X*. Moreover, *if $r : X \to X_0$ is a continuous retraction of X onto a maximal Kolmogorov subspace X_0 of X, then $r^{-1}(x) = \mathrm{MaxADSet}(x)$ for any point x of X belonging to X_0, where $\mathrm{MaxADSet}(x)$ is a unique maximal anti-discrete subset of X containing x* (see [5] for the precise definition of the set $\mathrm{MaxADSet}(x)$). The retraction r from the last theorem is defined uniquely, and it is denoted in the text by ,,Stone-retraction". It has the following two remarkable properties: r is open, i.e., sends open sets in X to open sets in X_0, and r is closed, i.e., sends closed sets in X to closed sets in X_0.

These results may be obtained by the methods described by R.H. Warren in [17].

MML Identifier: `TSP_2`.

The terminology and notation used here are introduced in the following articles: [15], [16], [12], [18], [2], [3], [14], [9], [19], [13], [6], [5], and [7].

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1. Maximal T_0-Subsets

Let X be a topological space. Let us observe that a subset of X is T_0 if:

(Def.1) For all points a, b of X such that $a \in$ it and $b \in$ it holds if $a \neq b$, then $\mathrm{MaxADSet}(a) \cap \mathrm{MaxADSet}(b) = \emptyset$.

Let X be a topological space. Let us observe that a subset of X is T_0 if:

(Def.2) For every point a of X such that $a \in$ it holds it $\cap\,\mathrm{MaxADSet}(a) = \{a\}$.

Let X be a topological space. Let us observe that a subset of X is T_0 if:

(Def.3) For every point a of X such that $a \in$ it there exists a subset D of X such that D is maximal anti-discrete and it $\cap D = \{a\}$.

Let Y be a topological structure. A subset of Y is maximal T_0 if:

(Def.4) It is T_0 and for every subset D of Y such that D is T_0 and it $\subseteq D$ holds it $= D$.

Next we state the proposition

(1) Let Y_0, Y_1 be topological structures, and let D_0 be a subset of Y_0, and let D_1 be a subset of Y_1. Suppose the topological structure of $Y_0 =$ the topological structure of Y_1 and $D_0 = D_1$. If D_0 is maximal T_0, then D_1 is maximal T_0.

Let X be a topological space. Let us observe that a subset of X is maximal T_0 if:

(Def.5) It is T_0 and $\mathrm{MaxADSet}(\mathrm{it}) =$ the carrier of X.

In the sequel X denotes a topological space.

We now state several propositions:

(2) For every subset M of X such that M is maximal T_0 holds M is dense.

(3) For every subset A of X such that A is open or closed holds if A is maximal T_0, then A is not proper.

(4) Every empty subset of X is not maximal T_0.

(5) Let M be a subset of X. Suppose M is maximal T_0. Let A be a subset of X. If A is closed, then $A = \mathrm{MaxADSet}(M \cap A)$.

(6) Let M be a subset of X. Suppose M is maximal T_0. Let A be a subset of X. If A is open, then $A = \mathrm{MaxADSet}(M \cap A)$.

(7) For every subset M of X such that M is maximal T_0 and for every subset A of X holds $\overline{A} = \mathrm{MaxADSet}(M \cap \overline{A})$.

(8) For every subset M of X such that M is maximal T_0 and for every subset A of X holds $\mathrm{Int}\,A = \mathrm{MaxADSet}(M \cap \mathrm{Int}\,A)$.

Let X be a topological space. Let us observe that a subset of X is maximal T_0 if:

(Def.6) For every point x of X there exists a point a of X such that $a \in$ it and it $\cap\,\mathrm{MaxADSet}(x) = \{a\}$.

The following two propositions are true:

(9) For every subset A of X such that A is T_0 there exists a subset M of X such that $A \subseteq M$ and M is maximal T_0.

(10) There exists subset of X which is maximal T_0.

2. Maximal Kolmogorov Subspaces

Let Y be a non empty topological structure. A subspace of Y is maximal T_0 if:

(Def.7) For every subset A of Y such that $A =$ the carrier of it holds A is maximal T_0.

One can prove the following proposition

(11) Let Y be a non empty topological structure, and let Y_0 be a subspace of Y, and let A be a subset of Y. Suppose $A =$ the carrier of Y_0. Then A is maximal T_0 if and only if Y_0 is maximal T_0.

Let Y be a non empty topological structure. Note that every subspace of Y which is maximal T_0 is also T_0 and every subspace of Y which is non T_0 is also non maximal T_0.

Let X be a topological space. Let us observe that a subspace of X is maximal T_0 if it satisfies the conditions (Def.8).

(Def.8) (i) It is T_0, and

(ii) for every T_0 subspace Y_0 of X such that it is a subspace of Y_0 holds the topological structure of it $=$ the topological structure of Y_0.

In the sequel X will be a topological space.

One can prove the following proposition

(12) Let A_0 be a non empty subset of X. Suppose A_0 is maximal T_0. Then there exists a strict subspace X_0 of X such that X_0 is maximal T_0 and $A_0 =$ the carrier of X_0.

Let X be a topological space. One can verify the following observations:

* every subspace of X which is maximal T_0 is also dense,
* every subspace of X which is non dense is also non maximal T_0,
* every subspace of X which is open and maximal T_0 is also non proper,
* every subspace of X which is open and proper is also non maximal T_0,
* every subspace of X which is proper and maximal T_0 is also non open,
* every subspace of X which is closed and maximal T_0 is also non proper,
* every subspace of X which is closed and proper is also non maximal T_0, and
* every subspace of X which is proper and maximal T_0 is also non closed.

Next we state the proposition

(13) Let Y_0 be a T_0 subspace of X. Then there exists a strict subspace X_0 of X such that Y_0 is a subspace of X_0 and X_0 is maximal T_0.

Let X be a topological space. Note that there exists a subspace of X which is maximal T_0 and strict.

Let X be a topological space. A maximal Kolmogorov subspace of X is a maximal T_0 subspace of X.

The following four propositions are true:

(14) Let X_0 be a maximal Kolmogorov subspace of X, and let G be a subset of X, and let G_0 be a subset of X_0. Suppose $G_0 = G$. Then G_0 is open if and only if the following conditions are satisfied:
  (i) MaxADSet(G) is open, and
  (ii) $G_0 = \text{MaxADSet}(G) \cap (\text{the carrier of } X_0)$.

(15) Let X_0 be a maximal Kolmogorov subspace of X and let G be a subset of X. Then G is open if and only if the following conditions are satisfied:
  (i) $G = \text{MaxADSet}(G)$, and
  (ii) there exists a subset G_0 of X_0 such that G_0 is open and $G_0 = G \cap (\text{the carrier of } X_0)$.

(16) Let X_0 be a maximal Kolmogorov subspace of X, and let F be a subset of X, and let F_0 be a subset of X_0. Suppose $F_0 = F$. Then F_0 is closed if and only if the following conditions are satisfied:
  (i) MaxADSet(F) is closed, and
  (ii) $F_0 = \text{MaxADSet}(F) \cap (\text{the carrier of } X_0)$.

(17) Let X_0 be a maximal Kolmogorov subspace of X and let F be a subset of X. Then F is closed if and only if the following conditions are satisfied:
  (i) $F = \text{MaxADSet}(F)$, and
  (ii) there exists a subset F_0 of X_0 such that F_0 is closed and $F_0 = F \cap (\text{the carrier of } X_0)$.

3. Stone Retraction Mapping Theorems

In the sequel X is a topological space and X_0 is a maximal Kolmogorov subspace of X.

One can prove the following propositions:

(18) Let r be a mapping from X into X_0 and let M be a subset of X. Suppose $M =$ the carrier of X_0. Suppose that for every point a of X holds $M \cap \text{MaxADSet}(a) = \{r(a)\}$. Then r is a continuous mapping from X into X_0.

(19) Let r be a mapping from X into X_0. Suppose that for every point a of X holds $r(a) \in \text{MaxADSet}(a)$. Then r is a continuous mapping from X into X_0.

(20) Let r be a continuous mapping from X into X_0 and let M be a subset of X. Suppose $M =$ the carrier of X_0. If for every point a of X holds $M \cap \text{MaxADSet}(a) = \{r(a)\}$, then r is a retraction.

(21) For every continuous mapping r from X into X_0 such that for every point a of X holds $r(a) \in \mathrm{MaxADSet}(a)$ holds r is a retraction.

(22) There exists continuous mapping from X into X_0 which is a retraction.

(23) X_0 is a retract of X.

Let X be a topological space and let X_0 be a maximal Kolmogorov subspace of X. Stone-retraction of X onto X_0 is a continuous mapping from X into X_0 and is defined as follows:

(Def.9) Stone-retraction of X onto X_0 is a retraction.

Next we state three propositions:

(24) Let a be a point of X and let b be a point of X_0. If $a = b$, then (Stone-retraction of X onto X_0) $^{-1}\,\overline{\{b\}} = \overline{\{a\}}$.

(25) For every point a of X and for every point b of X_0 such that $a = b$ holds (Stone-retraction of X onto X_0) $^{-1}\,\{b\} = \mathrm{MaxADSet}(a)$.

(26) For every subset E of X and for every subset F of X_0 such that $F = E$ holds (Stone-retraction of X onto X_0) $^{-1}\,F = \mathrm{MaxADSet}(E)$.

Let X be a topological space and let X_0 be a maximal Kolmogorov subspace of X. Then Stone-retraction of X onto X_0 is a continuous mapping from X into X_0 and it can be characterized by the condition:

(Def.10) Let M be a subset of X. Suppose $M =$ the carrier of X_0. Let a be a point of X. Then $M \cap \mathrm{MaxADSet}(a) = \{$(Stone-retraction of X onto $X_0)(a)\}$.

Let X be a topological space and let X_0 be a maximal Kolmogorov subspace of X. Then Stone-retraction of X onto X_0 is a continuous mapping from X into X_0 and it can be characterized by the condition:

(Def.11) For every point a of X holds (Stone-retraction of X onto $X_0)(a) \in \mathrm{MaxADSet}(a)$.

Next we state two propositions:

(27) For every point a of X holds (Stone-retraction of X onto X_0) $^{-1}$ $\{$(Stone-retraction of X onto $X_0)(a)\} = \mathrm{MaxADSet}(a)$.

(28) For every point a of X holds (Stone-retraction of X onto $X_0)^{\circ}\{a\} =$ (Stone-retraction of X onto $X_0)^{\circ}\,\mathrm{MaxADSet}(a)$.

Let X be a topological space and let X_0 be a maximal Kolmogorov subspace of X. Then Stone-retraction of X onto X_0 is a continuous mapping from X into X_0 and it can be characterized by the condition:

(Def.12) Let M be a subset of X. Suppose $M =$ the carrier of X_0. Let A be a subset of X. Then $M \cap \mathrm{MaxADSet}(A) =$ (Stone-retraction of X onto $X_0)^{\circ}A$.

The following propositions are true:

(29) For every subset A of X holds (Stone-retraction of X onto X_0) $^{-1}$ (Stone-retraction of X onto $X_0)^{\circ}A = \mathrm{MaxADSet}(A)$.

(30) For every subset A of X holds (Stone-retraction of X onto $X_0)^{\circ}A =$ (Stone-retraction of X onto $X_0)^{\circ}\,\mathrm{MaxADSet}(A)$.

(31) Let A, B be subsets of X. Then (Stone-retraction of X onto X_0)$^\circ(A \cup B)$ = (Stone-retraction of X onto X_0)$^\circ A \cup$ (Stone-retraction of X onto X_0)$^\circ B$.

(32) Let A, B be subsets of X. Suppose A is open or B is open. Then (Stone-retraction of X onto X_0)$^\circ(A \cap B)$ = (Stone-retraction of X onto X_0)$^\circ A \cap$ (Stone-retraction of X onto X_0)$^\circ B$.

(33) Let A, B be subsets of X. Suppose A is closed or B is closed. Then (Stone-retraction of X onto X_0)$^\circ(A \cap B)$ = (Stone-retraction of X onto X_0)$^\circ A \cap$ (Stone-retraction of X onto X_0)$^\circ B$.

(34) For every subset A of X such that A is open holds (Stone-retraction of X onto X_0)$^\circ A$ is open.

(35) For every subset A of X such that A is closed holds (Stone-retraction of X onto X_0)$^\circ A$ is closed.

References

[1] P. Alexandroff and H. H. Hopf. *Topologie I.* Springer-Verlag, Berlin, 1935.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Ryszard Engelking. *General Topology.* Volume 60 of *Monografie Matematyczne*, PWN - Polish Scientific Publishers, Warsaw, 1977.
[5] Zbigniew Karno. Maximal anti-discrete subspaces of topological spaces. *Formalized Mathematics*, 5(**1**):109–118, 1996.
[6] Zbigniew Karno. Maximal discrete subspaces of almost discrete topological spaces. *Formalized Mathematics*, 4(**1**):125–135, 1993.
[7] Zbigniew Karno. On Kolmogorov topological spaces. *Formalized Mathematics*, 5(**1**):119–124, 1996.
[8] Kazimierz Kuratowski. *Topology.* Volume I, PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.
[9] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[10] M. H. Stone. Application of boolean algebras to topology. *Math. Sb.*, 1:765–771, 1936.
[11] W.J. Thron. *Topological Structures.* Holt, Rinehart and Winston, New York, 1966.
[12] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[13] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[14] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[17] R.H. Warren. Identification spaces and unique uniformity. *Pacific Journal of Mathematics*, 95:483–492, 1981.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[19] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

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Projective Planes

Michał Muzalewski
Warsaw University
Białystok

Summary. The line of points a, b, denoted by $a \cdot b$ and the point of lines A, B denoted by $A \cdot B$ are defined. A few basic theorems related to these notions are proved. An inspiration for such approach comes from so called Leibniz program. Let us recall that the Leibniz program is a program of algebraization of geometry using purely geometric notions. Leibniz formulated his program in opposition to algebraization method developed by Descartes.

MML Identifier: PROJPL_1.

The terminology and notation used in this paper are introduced in the papers [2] and [1].

1. Projective Spaces

In this paper G will denote a projective incidence structure.

Let us consider G. A point of G is an element of the points of G. A line of G is an element of the lines of G.

We adopt the following rules: a, a_1, a_2, b, b_1, b_2, c, d, p, q, r will be points of G and A, B, M, N, P, Q, R will be lines of G.

Let us consider G, a, P. We introduce $a \nmid P$ as an antonym of $a \mid P$.

Let us consider G, a, b, P. The predicate $a, b \nmid P$ is defined as follows:

(Def.1) $a \nmid P$ and $b \nmid P$.

Let us consider G, a, P, Q. The predicate $a \mid P, Q$ is defined as follows:

(Def.2) $a \mid P$ and $a \mid Q$.

Let us consider G, a, P, Q, R. The predicate $a \mid P, Q, R$ is defined as follows:

(Def.3) $a \mid P$ and $a \mid Q$ and $a \mid R$.

We now state the proposition

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(1) (i) If $a, b \mid P$, then $b, a \mid P$,
(ii) if $a, b, c \mid P$, then $a, c, b \mid P$ and $b, a, c \mid P$ and $b, c, a \mid P$ and $c, a, b \mid P$ and $c, b, a \mid P$,
(iii) if $a \mid P, Q$, then $a \mid Q, P$, and
(iv) if $a \mid P, Q, R$, then $a \mid P, R, Q$ and $a \mid Q, P, R$ and $a \mid Q, R, P$ and $a \mid R, P, Q$ and $a \mid R, Q, P$.

A projective incidence structure is configuration if:

(Def.4) For all points p, q of it and for all lines P, Q of it such that $p \mid P$ and $q \mid P$ and $p \mid Q$ and $q \mid Q$ holds $p = q$ or $P = Q$.

We now state three propositions:

(2) G is configuration iff for all p, q, P, Q such that $p, q \mid P$ and $p, q \mid Q$ holds $p = q$ or $P = Q$.

(3) G is configuration iff for all p, q, P, Q such that $p \mid P, Q$ and $q \mid P, Q$ holds $p = q$ or $P = Q$.

(4) The following statements are equivalent
(i) G is a projective space defined in terms of incidence,
(ii) G is configuration and for all p, q there exists P such that $p, q \mid P$ and there exist p, P such that $p \nmid P$ and for every P there exist a, b, c such that a, b, c are mutually different and $a, b, c \mid P$ and for all a, b, c, d, p, M, N, P, Q such that $a, b, p \mid M$ and $c, d, p \mid N$ and $a, c \mid P$ and $b, d \mid Q$ and $p \nmid P$ and $p \nmid Q$ and $M \neq N$ there exists q such that $q \mid P, Q$.

An incidence projective plane is a 2-dimensional projective space defined in terms of incidence.

Let us consider G, a, b, c. We say that a, b and c are collinear if and only if:

(Def.5) There exists P such that $a, b, c \mid P$.

We introduce a, b, c form a triangle as an antonym of a, b and c are collinear.

Next we state two propositions:

(5) a, b and c are collinear iff there exists P such that $a \mid P$ and $b \mid P$ and $c \mid P$.

(6) a, b, c form a triangle iff for every P holds $a \nmid P$ or $b \nmid P$ or $c \nmid P$.

Let us consider G, a, b, c, d. We say that a, b, c, d form a quadrangle if and only if the conditions (Def.6) are satisfied.

(Def.6) (i) a, b, c form a triangle,
(ii) b, c, d form a triangle,
(iii) c, d, a form a triangle, and
(iv) d, a, b form a triangle.

One can prove the following propositions:

(7) If G is a projective space defined in terms of incidence, then there exist A, B such that $A \neq B$.

(8) Suppose G is a projective space defined in terms of incidence and $a \mid A$. Then there exist b, c such that $b, c \mid A$ and a, b, c are mutually different.

(9) Suppose G is a projective space defined in terms of incidence and $a \mid A$ and $A \neq B$. Then there exists b such that $b \mid A$ and $b \nmid B$ and $a \neq b$.

(10) If G is configuration and $a_1, a_2 \mid A$ and $a_1 \neq a_2$ and $b \nmid A$, then a_1, a_2, b form a triangle.

(11) Suppose a, b and c are collinear. Then
(i) a, c and b are collinear,
(ii) b, a and c are collinear,
(iii) b, c and a are collinear,
(iv) c, a and b are collinear, and
(v) c, b and a are collinear.

(12) Suppose a, b, c form a triangle. Then
(i) a, c, b form a triangle,
(ii) b, a, c form a triangle,
(iii) b, c, a form a triangle,
(iv) c, a, b form a triangle, and
(v) c, b, a form a triangle.

(13) Suppose a, b, c, d form a quadrangle. Then
(i) a, c, b, d form a quadrangle,
(ii) b, a, c, d form a quadrangle,
(iii) b, c, a, d form a quadrangle,
(iv) c, a, b, d form a quadrangle,
(v) c, b, a, d form a quadrangle,
(vi) a, b, d, c form a quadrangle,
(vii) a, c, d, b form a quadrangle,
(viii) b, a, d, c form a quadrangle,
(ix) b, c, d, a form a quadrangle,
(x) c, a, d, b form a quadrangle,
(xi) c, b, d, a form a quadrangle,
(xii) a, d, b, c form a quadrangle,
(xiii) a, d, c, b form a quadrangle,
(xiv) b, d, a, c form a quadrangle,
(xv) b, d, c, a form a quadrangle,
(xvi) c, d, a, b form a quadrangle,
(xvii) c, d, b, a form a quadrangle,
(xviii) d, a, b, c form a quadrangle,
(xix) d, a, c, b form a quadrangle,
(xx) d, b, a, c form a quadrangle,
(xxi) d, b, c, a form a quadrangle,
(xxii) d, c, a, b form a quadrangle, and
(xxiii) d, c, b, a form a quadrangle.

(14) If G is configuration and $a_1, a_2 \mid A$ and $b_1, b_2 \mid B$ and $a_1, a_2 \nmid B$ and $b_1, b_2 \nmid A$ and $a_1 \neq a_2$ and $b_1 \neq b_2$, then a_1, a_2, b_1, b_2 form a quadrangle.

(15) Suppose G is a projective space defined in terms of incidence. Then there exist a, b, c, d such that a, b, c, d form a quadrangle.

Let G be a projective space defined in terms of incidence. An element of [: the points of G, the points of G, the points of G, the points of G :] is called a quadrangle of G if:

(Def.7) $\mathrm{it}_{\mathbf{1}}$, $\mathrm{it}_{\mathbf{2}}$, $\mathrm{it}_{\mathbf{3}}$, $\mathrm{it}_{\mathbf{4}}$ form a quadrangle.

Let G be a projective space defined in terms of incidence and let a, b be points of G. Let us assume that $a \neq b$. The functor $a \cdot b$ yields a line of G and is defined as follows:

(Def.8) $a, b \mid a \cdot b$.

Next we state the proposition

(16) Let G be a projective space defined in terms of incidence, and let a, b be points of G, and let L be a line of G. Suppose $a \neq b$. Then $a \mid a \cdot b$ and $b \mid a \cdot b$ and $a \cdot b = b \cdot a$ and if $a \mid L$ and $b \mid L$, then $L = a \cdot b$.

2. Projective Planes

The following propositions are true:

(17) If there exist a, b, c such that a, b, c form a triangle and for all p, q there exists M such that $p, q \mid M$, then there exist p, P such that $p \nmid P$.

(18) If there exist a, b, c, d such that a, b, c, d form a quadrangle, then there exist a, b, c such that a, b, c form a triangle.

(19) If a, b, c form a triangle and $a, b \mid P$ and $a, c \mid Q$, then $P \neq Q$.

(20) If a, b, c, d form a quadrangle and $a, b \mid P$ and $a, c \mid Q$ and $a, d \mid R$, then P, Q, R are mutually different.

(21) Suppose G is configuration and $a \mid P, Q, R$ and P, Q, R are mutually different and $a \nmid A$ and $p \mid A, P$ and $q \mid A, Q$ and $r \mid A, R$. Then p, q, r are mutually different.

(22) Suppose that
- (i) G is configuration,
- (ii) for all p, q there exists M such that $p, q \mid M$,
- (iii) for all P, Q there exists a such that $a \mid P, Q$, and
- (iv) there exist a, b, c, d such that a, b, c, d form a quadrangle.

Given P. Then there exist a, b, c such that a, b, c are mutually different and $a, b, c \mid P$.

(23) G is an incidence projective plane if and only if the following conditions are satisfied:
- (i) G is configuration,
- (ii) for all p, q there exists M such that $p, q \mid M$,
- (iii) for all P, Q there exists a such that $a \mid P, Q$, and
- (iv) there exist a, b, c, d such that a, b, c, d form a quadrangle.

We adopt the following convention: G will denote an incidence projective plane, a, q will denote points of G, and A, B will denote lines of G.

Let us consider G, A, B. Let us assume that $A \neq B$. The functor $A \cdot B$ yields a point of G and is defined by:

(Def.9) $A \cdot B \mid A, B$.

Next we state two propositions:

(24) If $A \neq B$, then $A \cdot B \mid A$ and $A \cdot B \mid B$ and $A \cdot B = B \cdot A$ and if $a \mid A$ and $a \mid B$, then $a = A \cdot B$.

(25) If $A \neq B$ and $a \mid A$ and $q \nmid A$ and $a \neq A \cdot B$, then $q \cdot a \cdot B \mid B$ and $q \cdot a \cdot B \nmid A$.

3. Some Useful Propositions

We adopt the following convention: G denotes a projective space defined in terms of incidence and a, b, c, d denote points of G.

We now state two propositions:

(26) If a, b, c form a triangle, then a, b, c are mutually different.

(27) If a, b, c, d form a quadrangle, then a, b, c, d are mutually different.

In the sequel G will be an incidence projective plane.

One can prove the following propositions:

(28) For all points a, b, c, d of G such that $a \cdot c = b \cdot d$ holds $a = c$ or $b = d$ or $c = d$ or $a \cdot c = c \cdot d$.

(29) For all points a, b, c, d of G such that $a \cdot c = b \cdot d$ holds $a = c$ or $b = d$ or $c = d$ or $a \mid c \cdot d$.

(30) Let G be an incidence projective plane, and let e, m, m' be points of G, and let I be a line of G. If $m \mid I$ and $m' \mid I$ and $m \neq m'$ and $e \nmid I$, then $m \cdot e \neq m' \cdot e$ and $e \cdot m \neq e \cdot m'$.

(31) Let G be an incidence projective plane, and let e be a point of G, and let I, L_1, L_2 be lines of G. If $e \mid L_1$ and $e \mid L_2$ and $L_1 \neq L_2$ and $e \nmid I$, then $I \cdot L_1 \neq I \cdot L_2$ and $L_1 \cdot I \neq L_2 \cdot I$.

(32) Let G be a projective space defined in terms of incidence and let a, b, q, q_1 be points of G. If $q \mid a \cdot b$ and $q \mid a \cdot q_1$ and $q \neq a$ and $q_1 \neq a$ and $a \neq b$, then $q_1 \mid a \cdot b$.

(33) Let G be a projective space defined in terms of incidence and let a, b, c be points of G. If $c \mid a \cdot b$ and $a \neq c$, then $b \mid a \cdot c$.

(34) Let G be an incidence projective plane, and let q_1, q_2, r_1, r_2 be points of G, and let H be a line of G. If $r_1 \neq r_2$ and $r_1 \mid H$ and $r_2 \mid H$ and $q_1 \nmid H$ and $q_2 \nmid H$, then $q_1 \cdot r_1 \neq q_2 \cdot r_2$.

(35) For all points a, b, c of G such that $a \mid b \cdot c$ holds $a = c$ or $b = c$ or $b \mid c \cdot a$.

(36) For all points a, b, c of G such that $a \mid b \cdot c$ holds $b = a$ or $b = c$ or $c \mid b \cdot a$.

(37) Let e, x_1, x_2, p_1, p_2 be points of G and let H, I be lines of G. Suppose $x_1 \mid I$ and $x_2 \mid I$ and $e \mid H$ and $e \nmid I$ and $x_1 \neq x_2$ and $p_1 \neq e$ and $p_2 \neq e$ and $p_1 \mid e \cdot x_1$ and $p_2 \mid e \cdot x_2$. Then there exists a point r of G such that $r \mid p_1 \cdot p_2$ and $r \mid H$ and $r \neq e$.

References

[1] Wojciech Leończuk and Krzysztof Prażmowski. Incidence projective spaces. *Formalized Mathematics*, 2(**2**):225–232, 1991.

[2] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

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The Formalization of Simple Graphs

Yozo Toda
Information Processing Center
Chiba University

Summary. A graph is simple when

- it is non-directed,
- there is at most one edge between two vertices,
- there is no loop of length one.

A formalization of simple graphs is given from scratch. There is already an article [9], dealing with the similar subject. It is not used as a starting-point, because [9] formalizes directed non-empty graphs. Given a set of vertices, edge is defined as an (unordered) pair of different two vertices and graph as a pair of a set of vertices and a set of edges.

The following concepts are introduced:

- simple graph structure,
- the set of all simple graphs,
- equality relation on graphs.
- the notion of degrees of vertices; the number of edges connected to, or the number of adjacent vertices,
- the notion of subgraphs,
- path, cycle,
- complete and bipartite complete graphs,

Theorems proved in this articles include:

- the set of simple graphs satisfies a certain minimality condition,
- equivalence between two notions of degrees.

MML Identifier: `SGRAPH1`.

The terminology and notation used in this paper have been introduced in the following articles: [13], [1], [4], [6], [7], [2], [3], [8], [5], [11], [10], and [12].

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1. Preliminaries

Let m, n be natural numbers. The functor $[m,n]_{\mathbb{N}}$ yields a finite subset of $\mathbb{N}$ and is defined by:

(Def.1) $[m,n]_{\mathbb{N}} = \{i : i$ ranges over natural numbers, $m \leq i \ \wedge \ i \leq n\}$.

The following propositions are true:

(1) For all natural numbers m, n holds $[m,n]_{\mathbb{N}} = \{i : i$ ranges over natural numbers, $m \leq i \ \wedge \ i \leq n\}$.

(2) Let m, n be natural numbers and let e be arbitrary. Then $e \in [m,n]_{\mathbb{N}}$ if and only if there exists a natural number i such that $e = i$ and $m \leq i$ and $i \leq n$.

(3) For all natural numbers m, n, k holds $k \in [m,n]_{\mathbb{N}}$ iff $m \leq k$ and $k \leq n$.

(4) For every natural number n holds $[1,n]_{\mathbb{N}} = \operatorname{Seg} n$.

(5) For all natural numbers m, n such that $1 \leq m$ holds $[m,n]_{\mathbb{N}} \subseteq \operatorname{Seg} n$.

(6) For all natural numbers k, m, n such that $k < m$ holds $\operatorname{Seg} k \cap [m,n]_{\mathbb{N}} = \emptyset$.

(7) For all natural numbers m, n such that $n < m$ holds $[m,n]_{\mathbb{N}} = \emptyset$.

Let A, B be sets and let f be a function from A into B. We say that f is onto if and only if:

(Def.2) $\operatorname{rng} f = B$.

Let A, B be sets and let f be a function from A into B. We say that f is bijective if and only if:

(Def.3) f is one-to-one and onto.

One can prove the following proposition

(8) For every finite set z holds $\operatorname{card} z = 2$ iff there exist arbitrary x, y such that $x \in z$ and $y \in z$ and $x \neq y$ and $z = \{x, y\}$.

Let A be a set. The functor TwoElementSets(A) yields a set and is defined by:

(Def.4) TwoElementSets(A) $= \{z : z$ ranges over finite elements of 2^{A}, $\operatorname{card} z = 2\}$.

The following propositions are true:

(9) For every set A and for arbitrary e holds $e \in$ TwoElementSets(A) iff there exists a finite subset z of A such that $e = z$ and $\operatorname{card} z = 2$.

(10) Let A be a set and let e be arbitrary. Then $e \in$ TwoElementSets(A) if and only if the following conditions are satisfied:

(i) e is a finite subset of A, and

(ii) there exist arbitrary x, y such that $x \in A$ and $y \in A$ and $x \neq y$ and $e = \{x, y\}$.

(11) For every set A holds TwoElementSets(A) $\subseteq 2^{A}$.

(12) For every set A and for arbitrary e_1, e_2 such that $\{e_1, e_2\} \in$ TwoElementSets(A) holds $e_1 \in A$ and $e_2 \in A$ and $e_1 \neq e_2$.

(13) TwoElementSets($\emptyset$) $= \emptyset$.

(14) For all sets t, u such that $t \subseteq u$ holds TwoElementSets(t) $\subseteq$ TwoElementSets(u).

(15) For every finite set A holds TwoElementSets(A) is finite.

(16) For every non trivial set A holds TwoElementSets(A) is non empty.

(17) For arbitrary a holds TwoElementSets($\{a\}$) $= \emptyset$.

Let a be a set.

(Def.5) $\phi(a)$ is an empty subset of TwoElementSets(a).

Let X be an empty set. Observe that every subset of X is empty.

In the sequel X will be a set.

2. Simple Graphs

We introduce simple graph structures which are systems

⟨ SVertices, SEdges ⟩,

where the SVertices constitute a set and the SEdges constitute a subset of TwoElementSets(the SVertices).

Let X be a set. The functor SimpleGraphs(X) yields a non empty set and is defined as follows:

(Def.6) SimpleGraphs(X) $= \{\langle v, e\rangle : v$ ranges over finite subsets of X, e ranges over finite subsets of TwoElementSets(v)$\}$.

Next we state the proposition

(19)[1] $\langle \emptyset, \phi(\emptyset)\rangle \in$ SimpleGraphs(X).

Let X be a set. A strict simple graph structure is said to be a simple graph of X if:

(Def.7) It is an element of SimpleGraphs(X).

Next we state two propositions:

(20) SimpleGraphs(X) $= \{\langle v, e\rangle : v$ ranges over finite subsets of X, e ranges over finite subsets of TwoElementSets(v)$\}$.

(21) Let g be arbitrary. Then $g \in$ SimpleGraphs(X) if and only if there exists a finite subset v of X and there exists a finite subset e of TwoElementSets(v) such that $g = \langle v, e\rangle$.

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[1]The proposition (18) has been removed.

3. Equality Relation on Simple Graphs

One can prove the following propositions:

(23)[2] For every simple graph g of X holds the SVertices of $g \subseteq X$ and the SEdges of $g \subseteq$ TwoElementSets(the SVertices of g).

(24) For every simple graph g of X holds $g = \langle$the SVertices of g, the SEdges of $g\rangle$.

(25) Let g be a simple graph of X and let e be arbitrary. Suppose $e \in$ the SEdges of g. Then there exist arbitrary v_1, v_2 such that $v_1 \in$ the SVertices of g and $v_2 \in$ the SVertices of g and $v_1 \neq v_2$ and $e = \{v_1, v_2\}$.

(26) Let g be a simple graph of X and let v_1, v_2 be arbitrary. Suppose $\{v_1, v_2\} \in$ the SEdges of g. Then $v_1 \in$ the SVertices of g and $v_2 \in$ the SVertices of g and $v_1 \neq v_2$.

(27) Let g be a simple graph of X. Then
 (i) the SVertices of g is a finite subset of X, and
 (ii) the SEdges of g is a finite subset of TwoElementSets(the SVertices of g).

Let us consider X and let G, G' be simple graphs of X. We say that G is isomorphic to G' if and only if the condition (Def.8) is satisfied.

(Def.8) There exists a function F_1 from the SVertices of G into the SVertices of G' such that
 (i) F_1 is bijective, and
 (ii) for all elements v_1, v_2 of the SVertices of G holds $\{v_1, v_2\} \in$ the SEdges of G iff $\{F_1(v_1), F_1(v_2)\} \in$ the SEdges of G.

4. Properties of Simple Graphs

The scheme *IndSimpleGraphs0* concerns a set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For arbitrary G such that $G \in$ SimpleGraphs($\mathcal{A}$) holds $\mathcal{P}[G]$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[\langle\emptyset, \phi(\emptyset)\rangle]$,
- Let g be a simple graph of $\mathcal{A}$ and let v be arbitrary. Suppose $g \in$ SimpleGraphs($\mathcal{A}$) and $\mathcal{P}[g]$ and $v \in \mathcal{A}$ and $v \notin$ the SVertices of g. Then $\mathcal{P}[\langle(\text{the SVertices of } g) \cup \{v\}, \phi((\text{the SVertices of } g) \cup \{v\})\rangle]$,
- Let g be a simple graph of $\mathcal{A}$ and let e be arbitrary. Suppose $\mathcal{P}[g]$ and $e \in$ TwoElementSets(the SVertices of g) and $e \notin$ the SEdges of g. Then there exists a subset s_1 of TwoElementSets(the SVertices of g) such that $s_1 =$ (the SEdges of g) $\cup \{e\}$ and $\mathcal{P}[\langle$the SVertices of g, $s_1\rangle]$.

[2]The proposition (22) has been removed.

We now state three propositions:

(28) Let g be a simple graph of X. Then $g = \langle \emptyset, \phi(\emptyset) \rangle$ or there exists a set v and there exists a subset e of TwoElementSets(v) such that v is non empty and $g = \langle v, e \rangle$.

(30)[3] Let V be a subset of X, and let E be a subset of TwoElementSets(V), and let n be arbitrary, and let E_1 be a finite subset of TwoElementSets($V \cup \{n\}$). If $\langle V, E \rangle \in$ SimpleGraphs(X) and $n \in X$ and $n \notin V$, then $\langle V \cup \{n\}, E_1 \rangle \in$ SimpleGraphs(X).

(31) Let V be a subset of X, and let E be a subset of TwoElementSets(V), and let v_1, v_2 be arbitrary. Suppose $v_1 \in V$ and $v_2 \in V$ and $v_1 \neq v_2$ and $\langle V, E \rangle \in$ SimpleGraphs(X). Then there exists a finite subset v_3 of TwoElementSets(V) such that $v_3 = E \cup \{\{v_1, v_2\}\}$ and $\langle V, v_3 \rangle \in$ SimpleGraphs(X).

Let X be a set and let G_1 be a set. We say that G_1 is a set of simple graphs of X if and only if the conditions (Def.9) are satisfied.

(Def.9) (i) $\langle \emptyset, \phi(\emptyset) \rangle \in G_1$,

(ii) for every subset V of X and for every subset E of TwoElementSets(V) and for arbitrary n and for every finite subset E_1 of TwoElementSets($V \cup \{n\}$) such that $\langle V, E \rangle \in G_1$ and $n \in X$ and $n \notin V$ holds $\langle V \cup \{n\}, E_1 \rangle \in G_1$, and

(iii) for every subset V of X and for every subset E of TwoElementSets(V) and for arbitrary v_1, v_2 such that $\langle V, E \rangle \in G_1$ and $v_1 \in V$ and $v_2 \in V$ and $v_1 \neq v_2$ and $\{v_1, v_2\} \notin E$ there exists a finite subset v_3 of TwoElementSets(V) such that $v_3 = E \cup \{\{v_1, v_2\}\}$ and $\langle V, v_3 \rangle \in G_1$.

One can prove the following propositions:

(32) For arbitrary g_1 such that g_1 is a set of simple graphs of X holds $\langle \emptyset, \phi(\emptyset) \rangle \in g_1$.

(33) Let G_1 be arbitrary. Suppose G_1 is a set of simple graphs of X. Let V be a subset of X, and let E be a subset of TwoElementSets(V), and let n be arbitrary, and let E_1 be a finite subset of TwoElementSets($V \cup \{n\}$). If $\langle V, E \rangle \in G_1$ and $n \in X$ and $n \notin V$, then $\langle V \cup \{n\}, E_1 \rangle \in G_1$.

(34) Let G_1 be arbitrary. Suppose G_1 is a set of simple graphs of X. Let V be a subset of X, and let E be a subset of TwoElementSets(V), and let v_1, v_2 be arbitrary. Suppose $\langle V, E \rangle \in G_1$ and $v_1 \in V$ and $v_2 \in V$ and $v_1 \neq v_2$ and $\{v_1, v_2\} \notin E$. Then there exists a finite subset v_3 of TwoElementSets(V) such that $v_3 = E \cup \{\{v_1, v_2\}\}$ and $\langle V, v_3 \rangle \in G_1$.

(35) SimpleGraphs(X) is a set of simple graphs of X.

(36) For arbitrary O_1 such that O_1 is a set of simple graphs of X holds SimpleGraphs(X) $\subseteq O_1$.

(37) SimpleGraphs(X) is a set of simple graphs of X and for arbitrary O_1 such that O_1 is a set of simple graphs of X holds SimpleGraphs(X) $\subseteq O_1$.

[3]The proposition (29) has been removed.

5. Subgraphs

Let X be a set and let G be a simple graph of X. A simple graph of X is called a subgraph of G if:

(Def.10) The SVertices of it $\subseteq$ the SVertices of G and the SEdges of it $\subseteq$ the SEdges of G.

6. Degree of Vertices

Let X be a set, let G be a simple graph of X, and let v be arbitrary. Let us assume that $v \in$ the SVertices of G. The functor degree(G, v) yielding a natural number is defined by:

(Def.11) There exists a finite set X such that for arbitrary z holds $z \in X$ iff $z \in$ the SEdges of G and $v \in z$ and degree$(G, v) = \operatorname{card} X$.

One can prove the following propositions:

(38) Let G be a simple graph of X and let v be arbitrary. Suppose $v \in$ the SVertices of G. Then there exists a finite set Y such that for arbitrary z holds $z \in Y$ iff $z \in$ the SEdges of G and $v \in z$ and degree$(G, v) = \operatorname{card} Y$.

(39) Let X be a non empty set, and let G be a simple graph of X, and let v be arbitrary. Suppose $v \in$ the SVertices of G. Then there exists a finite set w_1 such that $w_1 = \{w : w$ ranges over elements of X, $w \in$ the SVertices of $G \wedge \{v, w\} \in$ the SEdges of $G\}$ and degree$(G, v) = \operatorname{card} w_1$.

(40) Let X be a non empty set, and let g be a simple graph of X, and let v be arbitrary. Suppose $v \in$ the SVertices of g. Then there exists a finite set V_1 such that $V_1 =$ the SVertices of g and degree$(g, v) < \operatorname{card} V_1$.

(41) Let g be a simple graph of X and let v, e be arbitrary. Suppose $v \in$ the SVertices of g and $e \in$ the SEdges of g and degree$(g, v) = 0$. Then $v \notin e$.

(42) Let g be a simple graph of X, and let v be arbitrary, and let v_4 be a finite set. Suppose $v_4 =$ the SVertices of g and $v \in v_4$ and $1 +$ degree$(g, v) = \operatorname{card} v_4$. Let w be an element of v_4. If $v \neq w$, then there exists arbitrary e such that $e \in$ the SEdges of g and $e = \{v, w\}$.

7. Path and Cycle

Let X be a set, let g be a simple graph of X, let v_1, v_2 be elements of the SVertices of g, and let p be a finite sequence of elements of the SVertices of g. We say that p is a path of v_1 and v_2 if and only if the conditions (Def.12) are satisfied.

(Def.12) (i) $p(1) = v_1$,

(ii) $p(\mathrm{len}\, p) = v_2$,

(iii) for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\, p$ holds $\{p(i), p(i+1)\} \in$ the SEdges of g, and

(iv) for all natural numbers i, j such that $1 \leq i$ and $i < \mathrm{len}\, p$ and $i < j$ and $j < \mathrm{len}\, p$ holds $p(i) \neq p(j)$ and $\{p(i), p(i+1)\} \neq \{p(j), p(j+1)\}$.

Let X be a set, let g be a simple graph of X, and let v_1, v_2 be elements of the SVertices of g. The functor $\mathrm{Paths}(v_1, v_2)$ yields a subset of (the SVertices of g)* and is defined by:

(Def.13) $\mathrm{Paths}(v_1, v_2) = \{s_2 :$ s_2 ranges over elements of (the SVertices of g)*, s_2 is a path of v_1 and $v_2\}$.

One can prove the following three propositions:

(43) Let g be a simple graph of X and let v_1, v_2 be elements of the SVertices of g. Then $\mathrm{Paths}(v_1, v_2) = \{s_2 :$ s_2 ranges over elements of (the SVertices of g)*, s_2 is a path of v_1 and $v_2\}$.

(44) Let g be a simple graph of X, and let v_1, v_2 be elements of the SVertices of g, and let e be arbitrary. Then $e \in \mathrm{Paths}(v_1, v_2)$ if and only if there exists an element s_2 of (the SVertices of g)* such that $e = s_2$ and s_2 is a path of v_1 and v_2.

(45) Let g be a simple graph of X, and let v_1, v_2 be elements of the SVertices of g, and let e be an element of (the SVertices of g)*. If e is a path of v_1 and v_2, then $e \in \mathrm{Paths}(v_1, v_2)$.

Let X be a set, let g be a simple graph of X, and let p be arbitrary. We say that p is a cycle of g if and only if:

(Def.14) There exists an element v of the SVertices of g such that $p \in \mathrm{Paths}(v, v)$.

8. Some Famous Graphs

Let n, m be natural numbers. The functor $\mathrm{K}_{m,n}$ yielding a simple graph of $\mathbb{N}$ is defined by the condition (Def.16).

(Def.16)[4] There exists a subset e_3 of TwoElementSets(Seg$(m+n)$) such that $e_3 = \{\{i, j\} :$ i ranges over elements of $\mathbb{N}$, j ranges over elements of $\mathbb{N}$, $i \in \mathrm{Seg}\, m \ \wedge\ j \in [m+1, m+n]_{\mathbb{N}}\}$ and $\mathrm{K}_{m,n} = \langle \mathrm{Seg}(m+n), e_3 \rangle$.

Let n be a natural number. The functor K_n yields a simple graph of $\mathbb{N}$ and is defined by the condition (Def.17).

(Def.17) There exists a finite subset e_3 of TwoElementSets(Seg n) such that $e_3 = \{\{i, j\} :$ i ranges over elements of $\mathbb{N}$, j ranges over elements of $\mathbb{N}$, $i \in \mathrm{Seg}\, n \ \wedge\ j \in \mathrm{Seg}\, n \ \wedge\ i \neq j\}$ and $\mathrm{K}_n = \langle \mathrm{Seg}\, n, e_3 \rangle$.

The simple graph TriangleGraph of $\mathbb{N}$ is defined by:

(Def.18) TriangleGraph $= \mathrm{K}_3$.

[4]The definition (Def.15) has been removed.

One can prove the following propositions:

(46) There exists a subset e_3 of TwoElementSets(Seg 3) such that $e_3 = \{\{1,2\},\{2,3\},\{3,1\}\}$ and TriangleGraph $= \langle \mathrm{Seg}\,3, e_3\rangle$.

(47) The SVertices of TriangleGraph $=$ Seg 3 and the SEdges of TriangleGraph $= \{\{1,2\},\{2,3\},\{3,1\}\}$.

(48) $\{1,2\} \in$ the SEdges of TriangleGraph and $\{2,3\} \in$ the SEdges of TriangleGraph and $\{3,1\} \in$ the SEdges of TriangleGraph.

(49) $\langle 1\rangle \frown \langle 2\rangle \frown \langle 3\rangle \frown \langle 1\rangle$ is a cycle of TriangleGraph.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.
[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[9] Krzysztof Hryniewiecki. Graphs. *Formalized Mathematics*, 2(**3**):365–370, 1991.
[10] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.
[11] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(**1**):187–190, 1990.
[12] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[13] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

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Solvable Groups

Katarzyna Zawadzka
Warsaw University
Białystok

Summary. The concept of solvable group is introduced. Some theorems concerning heirdom of solvability are proved.

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The articles [7], [13], [3], [4], [11], [6], [5], [2], [1], [9], [10], [8], and [12] provide the terminology and notation for this paper.

In this paper G denotes a group and i denotes a natural number.

A group is solvable if it satisfies the condition (Def.1).

(Def.1) There exists a finite sequence F of elements of $\mathrm{SubGr}\,\mathrm{it}$ such that
- (i) $\mathrm{len}\,F > 0$,
- (ii) $F(1) = \Omega_{\mathrm{it}}$,
- (iii) $F(\mathrm{len}\,F) = \{\mathbf{1}\}_{\mathrm{it}}$, and
- (iv) for every i such that $i \in \mathrm{dom}\,F$ and $i+1 \in \mathrm{dom}\,F$ and for all strict subgroups G_1, G_2 of it such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a strict normal subgroup of G_1 and for every normal subgroup N of G_1 such that $N = G_2$ holds ${}^{G_1}/_{N}$ is commutative.

One can check that there exists a group which is solvable and strict.

One can prove the following propositions:

(1) Let G be a strict group and let H, F_1, F_2 be strict subgroups of G. Suppose F_1 is a normal subgroup of F_2. Then $F_1 \cap H$ is a normal subgroup of $F_2 \cap H$.

(2) Let G be a strict group, and let F_2 be a strict subgroup of G, and let F_1 be a strict normal subgroup of F_2, and let a, b be elements of F_2. Then $a \cdot F_1 \cdot (b \cdot F_1) = (a \cdot b) \cdot F_1$.

(3) Let G be a strict group, and let H, F_2 be strict subgroups of G, and let F_1 be a strict normal subgroup of F_2, and let G_2 be a strict subgroup of G. Suppose $G_2 = H \cap F_2$. Let G_1 be a normal subgroup of G_2. Suppose

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$G_1 = H \cap F_1$. Then there exists a subgroup G_3 of ${}^{F_2}/_{F_1}$ such that ${}^{G_2}/_{G_1}$ and G_3 are isomorphic.

(4) Let G be a strict group, and let H, F_2 be strict subgroups of G, and let F_1 be a strict normal subgroup of F_2, and let G_2 be a strict subgroup of G. Suppose $G_2 = F_2 \cap H$. Let G_1 be a normal subgroup of G_2. Suppose $G_1 = F_1 \cap H$. Then there exists a subgroup G_3 of ${}^{F_2}/_{F_1}$ such that ${}^{G_2}/_{G_1}$ and G_3 are isomorphic.

(5) For every solvable strict group G holds every strict subgroup of G is solvable.

(6) Let G be a strict group. Given a finite sequence F of elements of $\operatorname{SubGr} G$ such that

(i) $\operatorname{len} F > 0$,

(ii) $F(1) = \Omega_G$,

(iii) $F(\operatorname{len} F) = \{\mathbf{1}\}_G$, and

(iv) for every i such that $i \in \operatorname{dom} F$ and $i + 1 \in \operatorname{dom} F$ and for all strict subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i + 1)$ holds G_2 is a strict normal subgroup of G_1 and for every normal subgroup N of G_1 such that $N = G_2$ holds ${}^{G_1}/_{N}$ is a cyclic group.

Then G is solvable.

(7) Every strict commutative group is strict and solvable.

Let G, H be strict groups, let g be a homomorphism from G to H, and let A be a subgroup of G. The functor $g \restriction A$ yielding a homomorphism from A to H is defined as follows:

(Def.2) $g \restriction A = g \restriction (\text{the carrier of } A)$.

Let G, H be strict groups, let g be a homomorphism from G to H, and let A be a subgroup of G. The functor $g^\circ A$ yields a strict subgroup of H and is defined as follows:

(Def.3) $g^\circ A = \operatorname{Im}(g \restriction A)$.

Next we state a number of propositions:

(8) Let G, H be strict groups, and let g be a homomorphism from G to H, and let A be a subgroup of G. Then $\operatorname{rng}(g \restriction A) = g^\circ(\text{the carrier of } A)$.

(9) Let G, H be strict groups, and let g be a homomorphism from G to H, and let A be a strict subgroup of G. Then the carrier of $g^\circ A = g^\circ(\text{the carrier of } A)$.

(10) Let G, H be strict groups, and let h be a homomorphism from G to H, and let A be a strict subgroup of G. Then $\operatorname{Im}(h \restriction A)$ is a strict subgroup of $\operatorname{Im} h$.

(11) Let G, H be strict groups, and let h be a homomorphism from G to H, and let A be a strict subgroup of G. Then $h^\circ A$ is a strict subgroup of $\operatorname{Im} h$.

(12) For all strict groups G, H and for every homomorphism h from G to H holds $h^\circ(\{\mathbf{1}\}_G) = \{\mathbf{1}\}_H$ and $h^\circ(\Omega_G) = \Omega_{\operatorname{Im} h}$.

(13) Let G, H be strict groups, and let h be a homomorphism from G to H, and let A, B be strict subgroups of G. If A is a subgroup of B, then $h^\circ A$ is a subgroup of $h^\circ B$.

(14) Let G, H be strict groups, and let h be a homomorphism from G to H, and let A be a strict subgroup of G, and let a be an element of G. Then $h(a) \cdot h^\circ A = h^\circ(a \cdot A)$ and $h^\circ A \cdot h(a) = h^\circ(A \cdot a)$.

(15) Let G, H be strict groups, and let h be a homomorphism from G to H, and let A, B be subsets of G. Then $h^\circ A \cdot h^\circ B = h^\circ(A \cdot B)$.

(16) Let G, H be strict groups, and let h be a homomorphism from G to H, and let A, B be strict subgroups of G. Suppose A is a strict normal subgroup of B. Then $h^\circ A$ is a strict normal subgroup of $h^\circ B$.

(17) Let G, H be strict groups and let h be a homomorphism from G to H. If G is a solvable group, then $\operatorname{Im} h$ is solvable.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[6] Dariusz Surowik. Cyclic groups and some of their properties - part I. *Formalized Mathematics*, 2(**5**):623–627, 1991.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[8] Wojciech A. Trybulec. Classes of conjugation. Normal subgroups. *Formalized Mathematics*, 1(**5**):955–962, 1990.

[9] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.

[10] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Formalized Mathematics*, 1(**5**):855–864, 1990.

[11] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[12] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(**4**):573–578, 1991.

[13] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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