Representation Theorem for Boolean Algebras

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The notation and terminology used in this paper are introduced in the following articles: [9], [7], [4], [5], [3], [10], [11], [8], [12], [1], [2], and [6].

In the sequel $T$ is a topological space, $X, Y$ are subsets of $T$, and $x$ is arbitrary.

Let $T$ be a topological space. The functor $\text{OpenClosedSet}(T)$ yielding a non empty family of subsets of the carrier of $T$ is defined as follows:

(Def.1) $\text{OpenClosedSet}(T) = \{x : x \text{ ranges over subsets of } T, x \text{ is open } \land x \text{ is closed}\}$.

The following propositions are true:

(1) If $x \in \text{OpenClosedSet}(T)$, then there exists $X$ such that $X = x$.

(2) If $X \in \text{OpenClosedSet}(T)$, then $X$ is open.

(3) If $X \in \text{OpenClosedSet}(T)$, then $X$ is closed.

(4) If $X$ is open and closed, then $X \in \text{OpenClosedSet}(T)$.

Let $X$ be a non empty set and let $t$ be a non empty family of subsets of $X$. We see that the element of $t$ is a subset of $X$.

In the sequel $x, y, z$ will denote elements of $\text{OpenClosedSet}(T)$.

Let us consider $T$ and let $C, D$ be elements of $\text{OpenClosedSet}(T)$. Then $C \cup D$ is an element of $\text{OpenClosedSet}(T)$.

Let us consider $T$ and let $C, D$ be elements of $\text{OpenClosedSet}(T)$. Then $C \cap D$ is an element of $\text{OpenClosedSet}(T)$.

Let us consider $T$. The functor $\text{join}(T)$ yielding a binary operation on $\text{OpenClosedSet}(T)$ is defined by:

(Def.2) For all elements $A, B$ of $\text{OpenClosedSet}(T)$ holds $(\text{join}(T))(A, B) = A \cup B$. 

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Let us consider $T$. The functor meet($T$) yields a binary operation on OpenClosedSet($T$) and is defined by:

(Def.3) For all elements $A$, $B$ of OpenClosedSet($T$) holds $(\text{meet}(T))(A, B) = A \cap B$.

We now state several propositions:

(5) Let $x$, $y$ be elements of the carrier of $\langle \text{OpenClosedSet}(T), \text{join}(T), \text{meet}(T) \rangle$ and let $x'$, $y'$ be elements of OpenClosedSet($T$). If $x = x'$ and $y = y'$, then $x \cup y = x' \cup y'$.

(6) Let $x$, $y$ be elements of the carrier of $\langle \text{OpenClosedSet}(T), \text{join}(T), \text{meet}(T) \rangle$ and let $x'$, $y'$ be elements of OpenClosedSet($T$). If $x = x'$ and $y = y'$, then $x \cap y = x' \cap y'$.

(7) $\emptyset_T$ is an element of OpenClosedSet($T$).

(8) $\Omega_T$ is an element of OpenClosedSet($T$).

(9) For every element $x$ of OpenClosedSet($T$) holds $x^c$ is an element of OpenClosedSet($T$).

(10) $\langle \text{OpenClosedSet}(T), \text{join}(T), \text{meet}(T) \rangle$ is a lattice.

Let $T$ be a topological space. The functor OpenClosedSetLatt($T$) yields a lattice and is defined by:

(Def.4) OpenClosedSetLatt($T$) = $\langle \text{OpenClosedSet}(T), \text{join}(T), \text{meet}(T) \rangle$.

Next we state two propositions:

(11) For every topological space $T$ and for all elements $x$, $y$ of the carrier of OpenClosedSetLatt($T$) holds $x \cup y = x \cup y$.

(12) For every topological space $T$ and for all elements $x$, $y$ of the carrier of OpenClosedSetLatt($T$) holds $x \cap y = x \cap y$.

We follow a convention: $a$, $b$, $c$ denote elements of the carrier of $\langle \text{OpenClosedSet}(T), \text{join}(T), \text{meet}(T) \rangle$ and $x$, $y$, $z$ denote elements of OpenClosedSet($T$).

The following propositions are true:

(13) The carrier of OpenClosedSetLatt($T$) = OpenClosedSet($T$).

(14) OpenClosedSetLatt($T$) is Boolean.

(15) $\Omega_T$ is an element of the carrier of OpenClosedSetLatt($T$).

(16) $\emptyset_T$ is an element of the carrier of OpenClosedSetLatt($T$).

One can check that there exists a Boolean lattice which is non trivial.

For simplicity we adopt the following convention: $L_1$, $L_2$ denote lattices, $a$, $p$, $q'$ denote elements of the carrier of $B_1$, $U_1$ denotes a filter of $B_1$, $B$ denotes a subset of the carrier of $B_1$, and $D$ denotes a non empty subset of the carrier of $B_1$.

Let us consider $B_1$. The functor ultraset($B_1$) yields a non empty subset of the carrier of $B_1$ and is defined by:

(Def.5) ultraset($B_1$) = $\{ F : F$ is ultrafilter $\}$.

Next we state two propositions:
(18) \( x \in \text{ultraset}(B_1) \) iff there exists \( U_1 \) such that \( U_1 = x \) and \( U_1 \) is ultrafilter.

(19) For every \( a \) holds \( \{ F : F \text{ is ultrafilter} \land a \in F \} \subseteq \text{ultraset}(B_1) \).

Let us consider \( B_1 \). The functor \( \text{UFilter}(B_1) \) yielding a function is defined as follows:

(Def.6) \( \text{dom } \text{UFilter}(B_1) = \) the carrier of \( B_1 \) and for every element \( a \) of the carrier of \( B_1 \) holds \( (\text{UFilter}(B_1))(a) = \{ U_1 : U_1 \text{ is ultrafilter} \land a \in U_1 \} \).

Next we state several propositions:

(20) \( x \in (\text{UFilter}(B_1))(a) \) iff there exists \( F \) such that \( F = x \) and \( F \) is ultrafilter and \( a \in F \).

(21) \( F \in (\text{UFilter}(B_1))(a) \) iff \( F \) is ultrafilter and \( a \in F \).

(22) For every \( F \) such that \( F \) is ultrafilter holds \( a \cup b \in F \) iff \( a \in F \) or \( b \in F \).

(23) \( (\text{UFilter}(B_1))(a \cap b) = (\text{UFilter}(B_1))(a) \cap (\text{UFilter}(B_1))(b) \).

(24) \( (\text{UFilter}(B_1))(a \cup b) = (\text{UFilter}(B_1))(a) \cup (\text{UFilter}(B_1))(b) \).

Let us consider \( B_1 \). Then \( \text{UFilter}(B_1) \) is a function from the carrier of \( B_1 \) into \( 2^{\text{ultraset}(B_1)} \).

Let us consider \( B_1 \). The functor \( \text{StoneR}(B_1) \) yielding a non empty set is defined as follows:

(Def.7) \( \text{StoneR}(B_1) = \text{rng } \text{UFilter}(B_1) \).

The following propositions are true:

(25) \( \text{StoneR}(B_1) \subseteq 2^{\text{ultraset}(B_1)} \).

(26) \( x \in \text{StoneR}(B_1) \) iff there exists \( a \) such that \( (\text{UFilter}(B_1))(a) = x \).

Let us consider \( B_1 \). The functor \( \text{StoneSpace}(B_1) \) yielding a strict topological space is defined by:

(Def.8) The carrier of \( \text{StoneSpace}(B_1) = \) ultraset\((B_1)\) and the topology of \( \text{StoneSpace}(B_1) = \{ \bigcup A : A \text{ ranges over subsets of } 2^{\text{ultraset}(B_1)}, A \subseteq \text{StoneR}(B_1) \} \).

One can prove the following two propositions:

(27) If \( F \) is ultrafilter and \( F \not\in (\text{UFilter}(B_1))(a) \), then \( a \not\in F \).

(28) \( \text{ultraset}(B_1) \setminus (\text{UFilter}(B_1))(a) = (\text{UFilter}(B_1))(a^c) \).

Let us consider \( B_1 \). The functor \( \text{StoneBLattice}(B_1) \) yields a lattice and is defined as follows:

(Def.9) \( \text{StoneBLattice}(B_1) = \text{OpenClosedSetLatt}(\text{StoneSpace}(B_1)) \).

One can prove the following four propositions:

(29) \( \text{UFilter}(B_1) \) is one-to-one.

(30) \( \bigcup \text{StoneR}(B_1) = \text{ultraset}(B_1) \).

(31) For all sets \( A, B, X \) such that \( X \subseteq \bigcup (A \cup B) \) and for arbitrary \( Y \) such that \( Y \in B \) holds \( Y \cap X = \emptyset \) holds \( X \subseteq \bigcup A \).

(32) For every non empty set \( X \) holds there exists finite subset of \( X \) which is non empty.

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1The proposition (17) has been removed.
Let $D$ be a non empty set. Note that there exists a finite subset of $D$ which is non empty.

The following propositions are true:

(33) For every lattice $L$ and for all elements $a, b, c, d$ of the carrier of $L$ such that $a \subseteq c$ and $b \subseteq d$ holds $a \cap b \subseteq c \cap d$.

(34) Let $L$ be a non trivial Boolean lattice and let $D$ be a non empty subset of the carrier of $L$. Suppose $\bot_L \in [D]$. Then there exists a non empty finite subset $B$ of the carrier of $L$ such that $B \subseteq D$ and $\bigcap_{B} = \bot_L$.

(35) For every lower bound lattice $L$ it is not true that there exists a filter $F$ of $L$ such that $F$ is ultrafilter and $\bot_L \in F$.

(36) $(\mathit{UF} \mathit{Filter}(B_1))(\bot(B_1)) = \emptyset$.

(37) $(\mathit{UF} \mathit{Filter}(B_1))(\top(B_1)) = \mathit{ultraset}(B_1)$.

(38) If ultraset$(B_1) = \bigcup X$ and $X$ is a subset of StoneR$(B_1)$, then there exists a finite subset $Y$ of $X$ such that ultraset$(B_1) = \bigcup Y$.

(39) If $x \in 2^X$ and $y \in 2^X$, then $x \cap y \in 2^X$.

(40) StoneR$(B_1) = \mathit{OpenClosedSet}(\mathit{StoneSpace}(B_1))$.

Let us consider $B_1$. Then $\mathit{UF} \mathit{Filter}(B_1)$ is a homomorphism from $B_1$ to StoneBLattice$(B_1)$.

Next we state four propositions:

(41) $\mathit{rng} \mathit{UF} \mathit{Filter}(B_1)$ = the carrier of StoneBLattice$(B_1)$.

(42) $\mathit{UF} \mathit{Filter}(B_1)$ is isomorphism.

(43) $B_1$ and StoneBLattice$(B_1)$ are isomorphic.

(44) For every non trivial Boolean lattice $B_1$ there exists a topological space $T$ such that $B_1$ and OpenClosedSetLatt$(T)$ are isomorphic.

REFERENCES


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