FORMALIZED MATHEMATICS
Volume 4, Number 1, 1993
Université Catholique de Louvain

Hahn-Banach Theorem

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Summary. We prove a version of Hahn-Banach Theorem.

MML Identifier: **HAHNBAN**.

The notation and terminology used here are introduced in the following papers: [13], [5], [9], [2], [3], [17], [16], [15], [8], [4], [10], [6], [14], [12], [11], [1], and [7].

1. Preliminaries

The following propositions are true:

(1) For arbitrary x, y and for every function f such that $\langle x, y\rangle \in f$ holds $y \in \operatorname{rng} f$.

(2) For every set X and for all functions f, g such that $X \subseteq \operatorname{dom} f$ and $f \subseteq g$ holds $f \restriction X = g \restriction X$.

(3) For every non empty set A and for arbitrary b such that $A \neq \{b\}$ there exists an element a of A such that $a \neq b$.

Let B be a non empty functional set. Observe that every element of B is function-like.

The following propositions are true:

(4) For all sets X, Y holds every non empty subset of $X \dot{\rightarrow} Y$ is a non empty functional set.

(5) Let B be a non empty functional set and let f be a function. Suppose $f = \bigcup B$. Then $\operatorname{dom} f = \bigcup\{\operatorname{dom} g : g \text{ ranges over elements of } B, \}$ and $\operatorname{rng} f = \bigcup\{\operatorname{rng} g : g \text{ ranges over elements of } B, \}$.

The scheme *NonUniqExD'* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

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ISSN 0777-4028

There exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element e of $\mathcal{A}$ holds $\mathcal{P}[e, f(e)]$

provided the parameters satisfy the following condition:

- For every element e of $\mathcal{A}$ there exists an element u of $\mathcal{B}$ such that $\mathcal{P}[e, u]$.

One can prove the following propositions:

(6) For every non empty subset A of $\overline{\mathbb{R}}$ such that for every *Real number* r such that $r \in A$ holds $r \leq -\infty$ holds $A = \{-\infty\}$.

(7) For every non empty subset A of $\overline{\mathbb{R}}$ such that for every *Real number* r such that $r \in A$ holds $+\infty \leq r$ holds $A = \{+\infty\}$.

(8) Let A be a non empty subset of $\overline{\mathbb{R}}$ and let r be a *Real number.* If $r < \sup A$, then there exists a *Real number* s such that $s \in A$ and $r < s$.

(9) Let A be a non empty subset of $\overline{\mathbb{R}}$ and let r be a *Real number.* If $\inf A < r$, then there exists a *Real number* s such that $s \in A$ and $s < r$.

(10) Let A, B be non empty subset of $\overline{\mathbb{R}}$. Suppose that for all *Real numbers* r, s such that $r \in A$ and $s \in B$ holds $r \leq s$. Then $\sup A \leq \inf B$.

(12)[1] Let x, y be real numbers and let x', y' be *Real numbers.* If $x = x'$ and $y = y'$, then $x \leq y$ iff $x' \leq y'$.

2. Sets Linearly Ordered by the Inclusion

A set is $\subseteq$ -linear if:

(Def.1) For arbitrary x, y such that $x \in$ it and $y \in$ it holds $x \subseteq y$ or $y \subseteq x$.

Let A be a non empty set. Note that there exists a subset of A which is $\subseteq$ -linear and non empty.

We now state the proposition

(13) For all sets X, Y and for every $\subseteq$ -linear non empty subset B of $X \dot{\to} Y$ holds $\bigcup B \in X \dot{\to} Y$.

3. Subspaces of a Real Linear Space

In the sequel V will be a real linear space.

One can prove the following propositions:

(14) For all subspaces W_1, W_2 of V holds the carrier of $W_1 \subseteq$ the carrier of $W_1 + W_2$.

(15) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$, then $v \lhd (W_1, W_2) = \langle v_1, v_2 \rangle$.

[1]The proposition (11) has been removed.

(16) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v \lhd (W_1, W_2) = \langle v_1, v_2 \rangle$, then $v = v_1 + v_2$.

(17) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v \lhd (W_1, W_2) = \langle v_1, v_2 \rangle$, then $v_1 \in W_1$ and $v_2 \in W_2$.

(18) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v \lhd (W_1, W_2) = \langle v_1, v_2 \rangle$, then $v \lhd (W_2, W_1) = \langle v_2, v_1 \rangle$.

(19) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_1$, then $v \lhd (W_1, W_2) = \langle v, 0_V \rangle$.

(20) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_2$, then $v \lhd (W_1, W_2) = \langle 0_V, v \rangle$.

(21) Let V_1 be a subspace of V, and let W_1 be a subspace of V_1, and let v be a vector of V. If $v \in W_1$, then v is a vector of V_1.

(22) For all subspaces V_1, V_2, W of V and for all subspaces W_1, W_2 of W such that $W_1 = V_1$ and $W_2 = V_2$ holds $W_1 + W_2 = V_1 + V_2$.

(23) For every subspace W of V and for every vector v of V and for every vector w of W such that $v = w$ holds $\mathrm{Lin}(\{w\}) = \mathrm{Lin}(\{v\})$.

(24) Let v be a vector of V and let X be a subspace of V. Suppose $v \notin X$. Let y be a vector of $X + \mathrm{Lin}(\{v\})$ and let W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $W = X$, then $X + \mathrm{Lin}(\{v\})$ is the direct sum of W and $\mathrm{Lin}(\{y\})$.

(25) Let v be a vector of V, and let X be a subspace of V, and let y be a vector of $X + \mathrm{Lin}(\{v\})$, and let W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $X = W$ and $v \notin X$, then $y \lhd (W, \mathrm{Lin}(\{y\})) = \langle 0_W, y \rangle$.

(26) Let v be a vector of V, and let X be a subspace of V, and let y be a vector of $X + \mathrm{Lin}(\{v\})$, and let W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. If $w \in X$, then $w \lhd (W, \mathrm{Lin}(\{y\})) = \langle w, 0_V \rangle$.

(27) For every vector v of V and for all subspaces W_1, W_2 of V there exist vectors v_1, v_2 of V such that $v \lhd (W_1, W_2) = \langle v_1, v_2 \rangle$.

(28) Let v be a vector of V, and let X be a subspace of V, and let y be a vector of $X + \mathrm{Lin}(\{v\})$, and let W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. Then there exists a vector x of X and there exists a real number r such that $w \lhd (W, \mathrm{Lin}(\{y\})) = \langle x, r \cdot v \rangle$.

(29) Let v be a vector of V, and let X be a subspace of V, and let y be a vector of $X + \mathrm{Lin}(\{v\})$, and let W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w_1, w_2 be vectors of $X + \mathrm{Lin}(\{v\})$, and let x_1, x_2 be vectors of X, and let r_1, r_2 be real numbers. If $w_1 \lhd (W, \mathrm{Lin}(\{y\})) = \langle x_1, r_1 \cdot v \rangle$ and $w_2 \lhd (W, \mathrm{Lin}(\{y\})) = \langle x_2, r_2 \cdot v \rangle$, then $(w_1 + w_2) \lhd (W, \mathrm{Lin}(\{y\})) = \langle x_1 + x_2, (r_1 + r_2) \cdot v \rangle$.

(30) Let v be a vector of V, and let X be a subspace of V, and let y be a vector of $X + \mathrm{Lin}(\{v\})$, and let W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$, and let x be a vector of X, and let t, r be real numbers. If $w \lhd (W, \mathrm{Lin}(\{y\})) = \langle x, r \cdot v \rangle$, then $(t \cdot w) \lhd (W, \mathrm{Lin}(\{y\})) = \langle t \cdot x, t \cdot r \cdot v \rangle$.

4. Functionals

Let V be an RLS structure.

(Def.2) A function from the carrier of V into $\mathbb{R}$ is called a functional in V.

Let us consider V. A functional in V is subadditive if:

(Def.3) For all vectors x, y of V holds $\mathrm{it}(x + y) \leq \mathrm{it}(x) + \mathrm{it}(y)$.

A functional in V is additive if:

(Def.4) For all vectors x, y of V holds $\mathrm{it}(x + y) = \mathrm{it}(x) + \mathrm{it}(y)$.

A functional in V is homogeneous if:

(Def.5) For every vector x of V and for every real number r holds $\mathrm{it}(r \cdot x) = r \cdot \mathrm{it}(x)$.

A functional in V is positively homogeneous if:

(Def.6) For every vector x of V and for every real number r such that $r > 0$ holds $\mathrm{it}(r \cdot x) = r \cdot \mathrm{it}(x)$.

A functional in V is semi-homogeneous if:

(Def.7) For every vector x of V and for every real number r such that $r \geq 0$ holds $\mathrm{it}(r \cdot x) = r \cdot \mathrm{it}(x)$.

A functional in V is absolutely homogeneous if:

(Def.8) For every vector x of V and for every real number r holds $\mathrm{it}(r \cdot x) = |r| \cdot \mathrm{it}(x)$.

A functional in V is 0-preserving if:

(Def.9) $\mathrm{It}(0_V) = 0$.

Let us consider V. One can verify the following observations:

* every functional in V which is additive is also subadditive,
* every functional in V which is homogeneous is also positively homogeneous,
* every functional in V which is semi-homogeneous is also positively homogeneous,
* every functional in V which is semi-homogeneous is also 0-preserving,
* every functional in V which is absolutely homogeneous is also semi-homogeneous, and
* every functional in V which is 0-preserving and positively homogeneous is also semi-homogeneous.

Let us consider V. Observe that there exists a functional in V which is additive absolutely homogeneous and homogeneous.

Let us consider V. A Banach functional in V is a subadditive positively homogeneous functional in V. A linear functional in V is an additive homogeneous functional in V.

We now state four propositions:

(31) For every homogeneous functional L in V and for every vector v of V holds $L(-v) = -L(v)$.

(32) For every linear functional L in V and for all vectors v_1, v_2 of V holds $L(v_1 - v_2) = L(v_1) - L(v_2)$.

(33) For every additive functional L in V holds $L(0_V) = 0$.

(34) Let X be a subspace of V, and let f_1 be a linear functional in X, and let v be a vector of V, and let y be a vector of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $v \notin X$. Let r be a real number. Then there exists a linear functional p_1 in $X + \mathrm{Lin}(\{v\})$ such that $p_1 \restriction$ (the carrier of X) $= f_1$ and $p_1(y) = r$.

5. Hahn-Banach Theorem

One can prove the following three propositions:

(35) Let V be a real linear space, and let X be a subspace of V, and let q be a Banach functional in V, and let f_1 be a linear functional in X. Suppose that for every vector x of X and for every vector v of V such that $x = v$ holds $f_1(x) \leq q(v)$. Then there exists a linear functional p_1 in V such that $p_1 \restriction$ (the carrier of X) $= f_1$ and for every vector x of V holds $p_1(x) \leq q(x)$.

(36) For every real normed space V holds the norm of V is an absolutely homogeneous subadditive functional in V.

(37) Let V be a real normed space, and let X be a subspace of V, and let f_1 be a linear functional in X. Suppose that for every vector x of X and for every vector v of V such that $x = v$ holds $f_1(x) \leq \|v\|$. Then there exists a linear functional p_1 in V such that $p_1 \restriction$ (the carrier of X) $= f_1$ and for every vector x of V holds $p_1(x) \leq \|x\|$.

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Received July 8, 1993