Hahn-Banach Theorem

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Summary. We prove a version of Hahn-Banach Theorem.

MML Identifier: HAHNBAN.

The notation and terminology used here are introduced in the following papers: [13], [5], [9], [2], [3], [17], [16], [15], [8], [4], [10], [6], [14], [12], [11], [1], and [7].

1. Preliminaries

The following propositions are true:
(1) For arbitrary \( x, y \) and for every function \( f \) such that \( (x, y) \in f \) holds \( y \in \text{rng } f \).
(2) For every set \( X \) and for all functions \( f, g \) such that \( X \subseteq \text{dom } f \) and \( f \subseteq g \) holds \( f \upharpoonright X = g \upharpoonright X \).
(3) For every non empty set \( A \) and for arbitrary \( b \) such that \( A \neq \{b\} \) there exists an element \( a \) of \( A \) such that \( a \neq b \).

Let \( B \) be a non empty functional set. Observe that every element of \( B \) is function-like.

The following propositions are true:
(4) For all sets \( X, Y \) holds every non empty subset of \( X \rightarrow Y \) is a non empty functional set.
(5) Let \( B \) be a non empty functional set and let \( f \) be a function. Suppose \( f = \bigcup B \). Then \( \text{dom } f = \bigcup \{ \text{dom } g : g \text{ ranges over elements of } B \} \) and \( \text{rng } f = \bigcup \{ \text{rng } g : g \text{ ranges over elements of } B \} \).

The scheme \( \text{NonUniqExD} \) deals with a non empty set \( A \), a non empty set \( B \), and a binary predicate \( \mathcal{P} \), and states that:
There exists a function $f$ from $\mathcal{A}$ into $\mathcal{B}$ such that for every element $e$ of $\mathcal{A}$ holds $\mathcal{P}[e, f(e)]$

provided the parameters satisfy the following condition:

- For every element $e$ of $\mathcal{A}$ there exists an element $u$ of $\mathcal{B}$ such that $\mathcal{P}[e, u]$.

One can prove the following propositions:

(6) For every non empty subset $A$ of $\mathbb{R}$ such that for every Real number $r$ such that $r \in A$ holds $r \leq -\infty$ holds $A = \{-\infty\}$.

(7) For every non empty subset $A$ of $\mathbb{R}$ such that for every Real number $r$ such that $r \in A$ holds $+\infty \leq r$ holds $A = \{+\infty\}$.

(8) Let $A$ be a non empty subset of $\mathbb{R}$ and let $r$ be a Real number. If $r < \sup A$, then there exists a Real number $s$ such that $s \in A$ and $r < s$.

(9) Let $A$ be a non empty subset of $\mathbb{R}$ and let $r$ be a Real number. If $\inf A < r$, then there exists a Real number $s$ such that $s \in A$ and $s < r$.

(10) Let $A, B$ be non empty subset of $\mathbb{R}$. Suppose that for all Real numbers $r, s$ such that $r \in A$ and $s \in B$ holds $r \leq s$. Then $\sup A \leq \inf B$.

(12) Let $x, y$ be real numbers and let $x', y'$ be Real numbers. If $x = x'$ and $y = y'$, then $x \leq y$ iff $x' \leq y'$.

2. Sets Linearly Ordered by the Inclusion

A set is $\subseteq$-linear if:

(Def.1) For arbitrary $x, y$ such that $x \in \textit{it}$ and $y \in \textit{it}$ holds $x \subseteq y$ or $y \subseteq x$.

Let $A$ be a non empty set. Note that there exists a subset of $A$ which is $\subseteq$-linear and non empty.

We now state the proposition

(13) For all sets $X, Y$ and for every $\subseteq$-linear non empty subset $B$ of $X \rightarrow Y$ holds $\bigcup B \in X \rightarrow Y$.

3. Subspaces of a Real Linear Space

In the sequel $V$ will be a real linear space.

One can prove the following propositions:

(14) For all subspaces $W_1, W_2$ of $V$ holds the carrier of $W_1 \subseteq$ the carrier of $W_1 + W_2$.

(15) Let $W_1, W_2$ be subspaces of $V$. Suppose $V$ is the direct sum of $W_1$ and $W_2$. Let $v, v_1, v_2$ be vectors of $V$. If $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$, then $v < (W_1, W_2) = \langle v_1, v_2 \rangle$.

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1The proposition (11) has been removed.
(16) Let \( W_1, W_2 \) be subspaces of \( V \). Suppose \( V \) is the direct sum of \( W_1 \) and \( W_2 \). Let \( v, v_1, v_2 \) be vectors of \( V \). If \( v \triangleleft (W_1, W_2) = \{v_1, v_2\} \), then \( v = v_1 + v_2 \).

(17) Let \( W_1, W_2 \) be subspaces of \( V \). Suppose \( V \) is the direct sum of \( W_1 \) and \( W_2 \). Let \( v, v_1, v_2 \) be vectors of \( V \). If \( v \triangleleft (W_1, W_2) = \{v_1, v_2\} \), then \( v_1 \in W_1 \) and \( v_2 \in W_2 \).

(18) Let \( W_1, W_2 \) be subspaces of \( V \). Suppose \( V \) is the direct sum of \( W_1 \) and \( W_2 \). Let \( v, v_1, v_2 \) be vectors of \( V \). If \( v \triangleleft (W_1, W_2) = \{v_1, v_2\} \), then \( v \triangleleft (W_2, W_1) = \{v_2, v_1\} \).

(19) Let \( W_1, W_2 \) be subspaces of \( V \). Suppose \( V \) is the direct sum of \( W_1 \) and \( W_2 \). Let \( v \) be a vector of \( V \). If \( v \in W_1 \), then \( v \triangleleft (W_1, W_2) = \{v, 0v\} \).

(20) Let \( W_1, W_2 \) be subspaces of \( V \). Suppose \( V \) is the direct sum of \( W_1 \) and \( W_2 \). Let \( v \) be a vector of \( V \). If \( v \in W_2 \), then \( v \triangleleft (W_1, W_2) = \{0v, v\} \).

(21) Let \( V \) be a subspace of \( V \), and let \( W_1 \) be a subspace of \( V \), and let \( v \) be a vector of \( V \). If \( v \in W_1 \), then \( v \) is a vector of \( V \).

(22) For all subspaces \( V_1, V_2, W \) of \( V \) and for all subspaces \( W_1, W_2 \) of \( W \) such that \( W_1 = V_1 \) and \( W_2 = V_2 \) holds \( W_1 + W_2 = V_1 + V_2 \).

(23) For every subspace \( W \) of \( V \) and for every vector \( v \) of \( W \) and for every vector \( w \) of \( W \) such that \( v = w \) holds \( \text{Lin}\{\{w\}\} = \text{Lin}\{\{v\}\} \).

(24) Let \( v \) be a vector of \( V \) and let \( X \) be a subspace of \( V \). Suppose \( v \notin X \). Let \( y \) be a vector of \( X + \text{Lin}\{\{v\}\} \) and let \( w \) be a subspace of \( X + \text{Lin}\{\{v\}\} \). If \( v = y \) and \( W = X \), then \( X + \text{Lin}\{\{v\}\} \) is the direct sum of \( W \) and \( \text{Lin}\{\{y\}\} \).

(25) Let \( v \) be a vector of \( V \), and let \( X \) be a subspace of \( V \), and let \( y \) be a vector of \( X + \text{Lin}\{\{v\}\} \), and let \( W \) be a subspace of \( X + \text{Lin}\{\{v\}\} \). If \( v = y \) and \( X = W \) and \( v \notin X \), then \( y \triangleleft (W, \text{Lin}\{\{y\}\}) = \{0w, y\} \).

(26) Let \( v \) be a vector of \( V \), and let \( X \) be a subspace of \( V \), and let \( y \) be a vector of \( X + \text{Lin}\{\{v\}\} \), and let \( W \) be a subspace of \( X + \text{Lin}\{\{v\}\} \). Suppose \( v = y \) and \( X = W \) and \( v \notin X \). Let \( w \) be a vector of \( X + \text{Lin}\{\{v\}\} \). If \( w \in X \), then \( w \triangleleft (W, \text{Lin}\{\{y\}\}) = \{w, 0v\} \).

(27) For every vector \( v \) of \( V \) and for all subspaces \( W_1, W_2 \) of \( V \) there exist vectors \( v_1, v_2 \) of \( V \) such that \( v \triangleleft (W_1, W_2) = \{v_1, v_2\} \).

(28) Let \( v \) be a vector of \( V \), and let \( X \) be a subspace of \( V \), and let \( y \) be a vector of \( X + \text{Lin}\{\{v\}\} \), and let \( W \) be a subspace of \( X + \text{Lin}\{\{v\}\} \). Suppose \( v = y \) and \( X = W \) and \( v \notin X \). Let \( w \) be a vector of \( X + \text{Lin}\{\{v\}\} \). Then there exists a vector \( z \) of \( X \) and there exists a real number \( r \) such that \( w \triangleleft (W, \text{Lin}\{\{y\}\}) = \{z, r \cdot v\} \).

(29) Let \( v \) be a vector of \( V \), and let \( X \) be a subspace of \( V \), and let \( y \) be a vector of \( X + \text{Lin}\{\{v\}\} \), and let \( W \) be a subspace of \( X + \text{Lin}\{\{v\}\} \). Suppose \( v = y \) and \( X = W \) and \( v \notin X \). Let \( w_1, w_2 \) be vectors of \( X + \text{Lin}\{\{v\}\} \), and let \( x_1, x_2 \) be vectors of \( X \), and let \( r_1, r_2 \) be real numbers. If \( w_1 \triangleleft (W, \text{Lin}\{\{y\}\}) = \{x_1, r_1 \cdot v\} \) and \( w_2 \triangleleft (W, \text{Lin}\{\{y\}\}) = \{x_2, r_2 \cdot v\} \), then \( (w_1 + w_2) \triangleleft (W, \text{Lin}\{\{y\}\}) = \{x_1 + x_2, (r_1 + r_2) \cdot v\} \).
(30) Let \( v \) be a vector of \( V \), and let \( X \) be a subspace of \( V \), and let \( y \) be a vector of \( X + \text{Lin}(\{v\}) \), and let \( W \) be a subspace of \( X + \text{Lin}(\{v\}) \). Suppose \( v = y \) and \( X = W \) and \( v \notin X \). Let \( w \) be a vector of \( X + \text{Lin}(\{v\}) \), and let \( x \) be a vector of \( X \), and let \( t, r \) be real numbers. If \( w \preceq (W, \text{Lin}(\{y\})) = (x, r \cdot v) \), then \((t \cdot w) \preceq (W, \text{Lin}(\{y\})) = (t \cdot x, t \cdot r \cdot v)\).

4. Functionals

Let \( V \) be an RLS structure.

(Def.2) A function from the carrier of \( V \) into \( \mathbb{R} \) is called a functional in \( V \).

Let us consider \( V \). A functional in \( V \) is subadditive if:

(Def.3) For all vectors \( x, y \) of \( V \) holds \( \text{it}(x + y) \leq \text{it}(x) + \text{it}(y) \).

A functional in \( V \) is additive if:

(Def.4) For all vectors \( x, y \) of \( V \) holds \( \text{it}(x + y) = \text{it}(x) + \text{it}(y) \).

A functional in \( V \) is homogeneous if:

(Def.5) For every vector \( x \) of \( V \) and for every real number \( r \) holds \( \text{it}(r \cdot x) = r \cdot \text{it}(x) \).

A functional in \( V \) is positively homogeneous if:

(Def.6) For every vector \( x \) of \( V \) and for every real number \( r \) such that \( r > 0 \) holds \( \text{it}(r \cdot x) = r \cdot \text{it}(x) \).

A functional in \( V \) is semi-homogeneous if:

(Def.7) For every vector \( x \) of \( V \) and for every real number \( r \) such that \( r \geq 0 \) holds \( \text{it}(r \cdot x) = r \cdot \text{it}(x) \).

A functional in \( V \) is absolutely homogeneous if:

(Def.8) For every vector \( x \) of \( V \) and for every real number \( r \) holds \( \text{it}(r \cdot x) = |r| \cdot \text{it}(x) \).

A functional in \( V \) is 0-preserving if:

(Def.9) \( \text{It}(0_V) = 0 \).

Let us consider \( V \). One can verify the following observations:

* every functional in \( V \) which is additive is also subadditive,
* every functional in \( V \) which is homogeneous is also positively homogeneous,
* every functional in \( V \) which is semi-homogeneous is also positively homogeneous,
* every functional in \( V \) which is semi-homogeneous is also 0-preserving,
* every functional in \( V \) which is absolutely homogeneous is also semi-homogeneous, and
* every functional in \( V \) which is 0-preserving and positively homogeneous is also semi-homogeneous.
Let us consider $V$. Observe that there exists a functional in $V$ which is additive absolutely homogeneous and homogeneous.


We now state four propositions:

(31) For every homogeneous functional $L$ in $V$ and for every vector $v$ of $V$ holds $L(-v) = -L(v)$.

(32) For every linear functional $L$ in $V$ and for all vectors $v_1$, $v_2$ of $V$ holds $L(v_1 - v_2) = L(v_1) - L(v_2)$.

(33) For every additive functional $L$ in $V$ holds $L(0_V) = 0$.

(34) Let $X$ be a subspace of $V$, and let $f_1$ be a linear functional in $X$, and let $v$ be a vector of $V$, and let $y$ be a vector of $X + \text{Lin}(\{v\})$. Suppose $v = y$ and $v \notin X$. Let $r$ be a real number. Then there exists a linear functional $p_1$ in $X + \text{Lin}(\{v\})$ such that $p_1 \upharpoonright (\text{the carrier of } X) = f_1$ and $p_1(y) = r$.

5. Hahn-Banach Theorem

One can prove the following three propositions:

(35) Let $V$ be a real linear space, and let $X$ be a subspace of $V$, and let $q$ be a Banach functional in $V$, and let $f_1$ be a linear functional in $X$. Suppose that for every vector $x$ of $X$ and for every vector $v$ of $V$ such that $x = v$ holds $f_1(x) \leq q(v)$. Then there exists a linear functional $p_1$ in $V$ such that $p_1 \upharpoonright (\text{the carrier of } X) = f_1$ and for every vector $x$ of $V$ holds $p_1(x) \leq q(x)$.

(36) For every real normed space $V$ holds the norm of $V$ is an absolutely homogeneous subadditive functional in $V$.

(37) Let $V$ be a real normed space, and let $X$ be a subspace of $V$, and let $f_1$ be a linear functional in $X$. Suppose that for every vector $x$ of $X$ and for every vector $v$ of $V$ such that $x = v$ holds $f_1(x) \leq \|v\|$. Then there exists a linear functional $p_1$ in $V$ such that $p_1 \upharpoonright (\text{the carrier of } X) = f_1$ and for every vector $x$ of $V$ holds $p_1(x) \leq \|x\|$.

References


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