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Cartesian Categories

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Summary. We define and prove some simple facts on Cartesian categories and its duals co-Cartesian categories. The Cartesian category is defined as a category with the fixed terminal object, the fixed projections, and the binary products. Category C has finite products if and only if C has a terminal object and for every pair a, b of objects of C the product $a \times b$ exists. We say that a category C has a finite product if every finite family of objects of C has a product. Our work is based on ideas of [13], where the algebraic properties of the proof theory are investigated. The terminal object of a Cartesian category C is denoted by $\mathbf{1}_C$. The binary product of a and b is written as $a \times b$. The projections of the product are written as $pr_1(a, b)$ and as $pr_2(a, b)$. We define the products $f \times g$ of arrows $f : a \to a'$ and $g : b \to b'$ as $< f \cdot pr_1, g \cdot pr_2 >: a \times b \to a' \times b'$

Co-Cartesian category is defined dually to the Cartesian category. Dual to a terminal object is an initial object, and to products are coproducts. The initial object of a Cartesian category C is written as $\mathbf{0}_C$. Binary coproduct of a and b is written as $a+b$. Injections of the coproduct are written as $in_1(a, b)$ and as $in_2(a, b)$.

MML Identifier: CAT_4.

The terminology and notation used in this paper are introduced in the following papers: [16], [15], [11], [4], [5], [14], [9], [12], [2], [1], [3], [7], [6], [8], and [10].

1. Preliminaries

In the sequel o, m, r will be arbitrary. We now define two new constructions. Let us consider o, m, r. $[\langle o, m\rangle \mapsto r]$ is a function from $[\!:\{o\}, \{m\}:\!]$ into $\{r\}$.

Let C be a category, and let a, b be objects of C. Let us observe that a and b are isomorphic if:

(Def.1) $\hom(a, b) \neq \emptyset$ and $\hom(b, a) \neq \emptyset$ and there exists a morphism f from a to b and there exists a morphism f' from b to a such that $f \cdot f' = \mathrm{id}_b$ and $f' \cdot f = \mathrm{id}_a$.

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2. Cartesian Categories

Let C be a category. We say that C has finite product if and only if:

(Def.2) for every set I and for every function F from I into the objects of C such that I is finite there exists an object a of C and there exists a projections family F' from a onto I such that $\operatorname{cod}_\kappa F'(\kappa) = F$ and a is a product w.r.t. F'.

We now state the proposition

(1) Let C be a category. Then C has finite product if and only if there exists an object of C which is a terminal object and for every objects a, b of C there exists an object c of C and there exist morphisms p_1, p_2 of C such that $\operatorname{dom} p_1 = c$ and $\operatorname{dom} p_2 = c$ and $\operatorname{cod} p_1 = a$ and $\operatorname{cod} p_2 = b$ and c is a product w.r.t. p_1 and p_2.

We now define several new constructions. We consider Cartesian category structures which are extension of category structures and are systems

⟨objects, morphisms, a dom-map, a cod-map, a composition, an id-map, a terminal, a product, a 1st-projection, a 2nd-projection⟩,

where the objects, the morphisms constitute non-empty sets, the dom-map, the cod-map are functions from the morphisms into the objects, the composition is a partial function from $[\!:\text{the morphisms}, \text{the morphisms}:\!]$ to the morphisms, the id-map is a function from the objects into the morphisms, the terminal is an element of the objects, the product is a function from $[\!:\text{the objects}, \text{the objects}:\!]$ into the objects, and the 1st-projection, the 2nd-projection are functions from $[\!:\text{the objects}, \text{the objects}:\!]$ into the morphisms. Let C be a Cartesian category structure. The functor $\mathbf{1}_C$ yielding an object of C is defined by:

(Def.3) $\mathbf{1}_C =$ the terminal of C.

Let a, b be objects of C. The functor $a \times b$ yielding an object of C is defined as follows:

(Def.4) $a \times b =$ (the product of C)($\langle\!\langle a, b \rangle\!\rangle$).

The functor $\pi_1(a \times b)$ yielding a morphism of C is defined as follows:

(Def.5) $\pi_1(a \times b) =$ (the 1st-projection of C)($\langle\!\langle a, b \rangle\!\rangle$).

The functor $\pi_2(a \times b)$ yields a morphism of C and is defined as follows:

(Def.6) $\pi_2(a \times b) =$ (the 2nd-projection of C)($\langle\!\langle a, b \rangle\!\rangle$).

Let us consider o, m. The functor $\dot{\circlearrowleft}_{\rm c}(o, m)$ yielding a strict Cartesian category structure is defined by:

(Def.7) $\dot{\circlearrowleft}_{\rm c}(o, m) = \langle \{o\}, \{m\}, \{m\} \longmapsto o, \{m\} \longmapsto o, \langle m, m\rangle \longmapsto m, \{o\} \longmapsto m, \mathrm{Extract}(o), [\langle o, o\rangle \mapsto o], [\langle o, o\rangle \mapsto m], [\langle o, o\rangle \mapsto m]\rangle$.

We now state the proposition

(2) The category structure of $\dot{\circlearrowleft}_{\rm c}(o, m) = \dot{\circlearrowleft}(o, m)$.

Let us note that there exists a Cartesian category structure which is strict and category-like.

Let o, m be arbitrary. Then $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ is a strict category-like Cartesian category structure.

The following propositions are true:

(3) For every object a of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds $a = o$.

(4) For all objects a, b of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds $a = b$.

(5) For every morphism f of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds $f = m$.

(6) For all morphisms f, g of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds $f = g$.

(7) For all objects a, b of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ and for every morphism f of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds $f \in \hom(a, b)$.

(8) For all objects a, b of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ every morphism of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ is a morphism from a to b.

(9) For all objects a, b of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds $\hom(a, b) \neq \emptyset$.

(10) Every object of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ is a terminal object.

(11) For every object c of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ and for all morphisms p_1, p_2 of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds c is a product w.r.t. p_1 and p_2.

A category-like Cartesian category structure is Cartesian if:

(Def.8) the terminal of it is a terminal object and for all objects a, b of it holds cod (the 1st-projection of it)($\langle a, b \rangle$) $= a$ and cod (the 2nd-projection of it)($\langle a, b \rangle$) $= b$ and (the product of it)($\langle a, b \rangle$) is a product w.r.t. (the 1st-projection of it)($\langle a, b \rangle$) and (the 2nd-projection of it)($\langle a, b \rangle$).

We now state the proposition

(12) For arbitrary o, m holds $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ is Cartesian.

One can verify that there exists a strict Cartesian category-like Cartesian category structure.

A Cartesian category is a category-like Cartesian category structure.

We adopt the following convention: C denotes a Cartesian category and a, b, c, d, e, s denote objects of C. We now state three propositions:

(13) $\mathbf{1}_C$ is a terminal object.

(14) For all morphisms f_1, f_2 from a to $\mathbf{1}_C$ holds $f_1 = f_2$.

(15) $\hom(a, \mathbf{1}_C) \neq \emptyset$.

Let us consider C, a. $!_a$ is a morphism from a to $\mathbf{1}_C$.

Next we state several propositions:

(16) $!_a = |_{\mathbf{1}_C} a$.

(17) $\mathrm{dom}(!_a) = a$ and $\mathrm{cod}(!_a) = \mathbf{1}_C$.

(18) $\hom(a, \mathbf{1}_C) = \{!_a\}$.

(19) $\mathrm{dom}\, \pi_1(a \times b) = a \times b$ and $\mathrm{cod}\, \pi_1(a \times b) = a$.

(20) $\mathrm{dom}\, \pi_2(a \times b) = a \times b$ and $\mathrm{cod}\, \pi_2(a \times b) = b$.

Let us consider C, a, b. Then $\pi_1(a \times b)$ is a morphism from $a \times b$ to a. Then $\pi_2(a \times b)$ is a morphism from $a \times b$ to b.

The following four propositions are true:

(21) $\hom(a \times b, a) \neq \emptyset$ and $\hom(a \times b, b) \neq \emptyset$.

(22) $a \times b$ is a product w.r.t. $\pi_1(a \times b)$ and $\pi_2(a \times b)$.

(23) C has finite product.

(24) If $\hom(a, b) \neq \emptyset$ and $\hom(b, a) \neq \emptyset$, then $\pi_1(a \times b)$ is retraction and $\pi_2(a \times b)$ is retraction.

Let us consider C, a, b, c, and let f be a morphism from c to a, and let g be a morphism from c to b. Let us assume that $\hom(c, a) \neq \emptyset$ and $\hom(c, b) \neq \emptyset$. The functor $\langle f, g \rangle$ yields a morphism from c to $a \times b$ and is defined by:

(Def.9) $\pi_1(a \times b) \cdot \langle f, g \rangle = f$ and $\pi_2(a \times b) \cdot \langle f, g \rangle = g$.

The following propositions are true:

(25) If $\hom(c, a) \neq \emptyset$ and $\hom(c, b) \neq \emptyset$, then $\hom(c, a \times b) \neq \emptyset$.

(26) $\langle \pi_1(a \times b), \pi_2(a \times b) \rangle = \mathrm{id}_{(a \times b)}$.

(27) For every morphism f from c to a and for every morphism g from c to b and for every morphism h from d to c such that $\hom(c, a) \neq \emptyset$ and $\hom(c, b) \neq \emptyset$ and $\hom(d, c) \neq \emptyset$ holds $\langle f \cdot h, g \cdot h \rangle = \langle f, g \rangle \cdot h$.

(28) For all morphisms f, k from c to a and for all morphisms g, h from c to b such that $\hom(c, a) \neq \emptyset$ and $\hom(c, b) \neq \emptyset$ and $\langle f, g \rangle = \langle k, h \rangle$ holds $f = k$ and $g = h$.

(29) For every morphism f from c to a and for every morphism g from c to b such that $\hom(c, a) \neq \emptyset$ and $\hom(c, b) \neq \emptyset$ and also f is monic or g is monic holds $\langle f, g \rangle$ is monic.

(30) $\hom(a, a \times \mathbf{1}_C) \neq \emptyset$ and $\hom(a, \mathbf{1}_C \times a) \neq \emptyset$.

We now define four new functors. Let us consider C, a. The functor $\lambda(a)$ yielding a morphism from $\mathbf{1}_C \times a$ to a is defined by:

(Def.10) $\lambda(a) = \pi_2(\mathbf{1}_C \times a)$.

The functor $\lambda^{-1}(a)$ yielding a morphism from a to $\mathbf{1}_C \times a$ is defined as follows:

(Def.11) $\lambda^{-1}(a) = \langle !_a, \mathrm{id}_a \rangle$.

The functor $\rho(a)$ yields a morphism from $a \times \mathbf{1}_C$ to a and is defined as follows:

(Def.12) $\rho(a) = \pi_1(a \times \mathbf{1}_C)$.

The functor $\rho^{-1}(a)$ yielding a morphism from a to $a \times \mathbf{1}_C$ is defined as follows:

(Def.13) $\rho^{-1}(a) = \langle \mathrm{id}_a, !_a \rangle$.

The following propositions are true:

(31) $\lambda(a) \cdot \lambda^{-1}(a) = \mathrm{id}_a$ and $\lambda^{-1}(a) \cdot \lambda(a) = \mathrm{id}_{(\mathbf{1}_C \times a)}$ and $\rho(a) \cdot \rho^{-1}(a) = \mathrm{id}_a$ and $\rho^{-1}(a) \cdot \rho(a) = \mathrm{id}_{(a \times \mathbf{1}_C)}$.

(32) a and $a \times \mathbf{1}_C$ are isomorphic and a and $\mathbf{1}_C \times a$ are isomorphic.

Let us consider C, a, b. The functor $\mathrm{Switch}(a)$ yielding a morphism from $a \times b$ to $b \times a$ is defined as follows:

(Def.14) $\mathrm{Switch}(a) = \langle \pi_2(a \times b), \pi_1(a \times b) \rangle$.

One can prove the following three propositions:

(33) $\hom(a \times b, b \times a) \neq \emptyset$.

(34) $\text{Switch}(a) \cdot \text{Switch}(b) = \text{id}_{(b \times a)}$.

(35) $a \times b$ and $b \times a$ are isomorphic.

Let us consider C, a. The functor $\Delta(a)$ yielding a morphism from a to $a \times a$ is defined by:

(Def.15) $\Delta(a) = \langle \text{id}_a, \text{id}_a \rangle$.

We now state two propositions:

(36) $\text{hom}(a, a \times a) \neq \emptyset$.

(37) For every morphism f from a to b such that $\text{hom}(a, b) \neq \emptyset$ holds $\langle f, f \rangle = \Delta(b) \cdot f$.

We now define two new functors. Let us consider C, a, b, c. The functor $\alpha((a, b), c)$ yielding a morphism from $a \times b \times c$ to $a \times (b \times c)$ is defined by:

(Def.16) $\alpha((a, b), c) = \langle \pi_1(a \times b) \cdot \pi_1((a \times b) \times c), \langle \pi_2(a \times b) \cdot \pi_1((a \times b) \times c), \pi_2((a \times b) \times c) \rangle \rangle$.

The functor $\alpha(a, (b, c))$ yields a morphism from $a \times (b \times c)$ to $a \times b \times c$ and is defined as follows:

(Def.17) $\alpha(a, (b, c)) = \langle \langle \pi_1(a \times (b \times c)), \pi_1(b \times c) \cdot \pi_2(a \times (b \times c)) \rangle, \pi_2(b \times c) \cdot \pi_2(a \times (b \times c)) \rangle$.

The following three propositions are true:

(38) $\text{hom}(a \times b \times c, a \times (b \times c)) \neq \emptyset$ and $\text{hom}(a \times (b \times c), a \times b \times c) \neq \emptyset$.

(39) $\alpha((a, b), c) \cdot \alpha(a, (b, c)) = \text{id}_{(a \times (b \times c))}$ and $\alpha(a, (b, c)) \cdot \alpha((a, b), c) = \text{id}_{(a \times b \times c)}$.

(40) $(a \times b) \times c$ and $a \times (b \times c)$ are isomorphic.

Let us consider C, a, b, c, d, and let f be a morphism from a to b, and let g be a morphism from c to d. The functor $f \times g$ yields a morphism from $a \times c$ to $b \times d$ and is defined by:

(Def.18) $f \times g = \langle f \cdot \pi_1(a \times c), g \cdot \pi_2(a \times c) \rangle$.

One can prove the following propositions:

(41) If $\text{hom}(a, c) \neq \emptyset$ and $\text{hom}(b, d) \neq \emptyset$, then $\text{hom}(a \times b, c \times d) \neq \emptyset$.

(42) $\text{id}_a \times \text{id}_b = \text{id}_{(a \times b)}$.

(43) Let f be a morphism from a to b. Let h be a morphism from c to d. Then for every morphism g from e to a and for every morphism k from e to c such that $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(c, d) \neq \emptyset$ and $\text{hom}(e, a) \neq \emptyset$ and $\text{hom}(e, c) \neq \emptyset$ holds $(f \times h) \cdot \langle g, k \rangle = \langle f \cdot g, h \cdot k \rangle$.

(44) For every morphism f from c to a and for every morphism g from c to b such that $\text{hom}(c, a) \neq \emptyset$ and $\text{hom}(c, b) \neq \emptyset$ holds $\langle f, g \rangle = (f \times g) \cdot \Delta(c)$.

(45) Let f be a morphism from a to b. Let h be a morphism from c to d. Then for every morphism g from e to a and for every morphism k from s to c such that $\text{hom}(a, b) \neq \emptyset$ and $\text{hom}(c, d) \neq \emptyset$ and $\text{hom}(e, a) \neq \emptyset$ and $\text{hom}(s, c) \neq \emptyset$ holds $(f \times h) \cdot (g \times k) = (f \cdot g) \times (h \cdot k)$.

3. Co-Cartesian Categories

Let C be a category. We say that C has finite coproduct if and only if:

(Def.19) for every set I and for every function F from I into the objects of C such that I is finite there exists an object a of C and there exists a injections family F' into a on I such that $\mathrm{dom}_\kappa F'(\kappa) = F$ and a is a coproduct w.r.t. F'.

Next we state the proposition

(46) Let C be a category. Then C has finite coproduct if and only if there exists an object of C which is an initial object and for every objects a, b of C there exists an object c of C and there exist morphisms i_1, i_2 of C such that $\mathrm{dom}\, i_1 = a$ and $\mathrm{dom}\, i_2 = b$ and $\mathrm{cod}\, i_1 = c$ and $\mathrm{cod}\, i_2 = c$ and c is a coproduct w.r.t. i_1 and i_2.

We now define several new constructions. We consider cocartesian category structures which are extension of category structures and are systems

⟨objects, morphisms, a dom-map, a cod-map, a composition, an id-map, a initial, a coproduct, a 1st-coprojection, a 2nd-coprojection⟩,

where the objects, the morphisms constitute non-empty sets, the dom-map, the cod-map are functions from the morphisms into the objects, the composition is a partial function from $[\!:\text{the morphisms}, \text{the morphisms}:\!]$ to the morphisms, the id-map is a function from the objects into the morphisms, the initial is an element of the objects, the coproduct is a function from $[\!:\text{the objects}, \text{the objects}:\!]$ into the objects, and the 1st-coprojection, the 2nd-coprojection are functions from $[\!:\text{the objects}, \text{the objects}:\!]$ into the morphisms. Let C be a cocartesian category structure. The functor $\mathbf{0}_C$ yields an object of C and is defined as follows:

(Def.20) $\mathbf{0}_C =$ the initial of C.

Let a, b be objects of C. The functor $a + b$ yields an object of C and is defined as follows:

(Def.21) $a + b =$ (the coproduct of C)($\langle a, b \rangle$).

The functor $\mathrm{in}_1(a + b)$ yields a morphism of C and is defined as follows:

(Def.22) $\mathrm{in}_1(a + b) =$ (the 1st-coprojection of C)($\langle a, b \rangle$).

The functor $\mathrm{in}_2(a + b)$ yields a morphism of C and is defined by:

(Def.23) $\mathrm{in}_2(a + b) =$ (the 2nd-coprojection of C)($\langle a, b \rangle$).

Let us consider o, m. The functor $\dot{\circlearrowleft}_{\mathrm{c}}^{\mathrm{op}}(o, m)$ yielding a strict cocartesian category structure is defined by:

(Def.24) $\dot{\circlearrowleft}_{\mathrm{c}}^{\mathrm{op}}(o, m) = \langle \{o\}, \{m\}, \{m\} \longmapsto o, \{m\} \longmapsto o, \langle m, m \rangle \longmapsto m, \{o\} \longmapsto m, \mathrm{Extract}(o), [\langle o, o \rangle \mapsto o], [\langle o, o \rangle \mapsto m], [\langle o, o \rangle \mapsto m] \rangle$.

One can prove the following proposition

(47) The category structure of $\dot{\circlearrowleft}_{\mathrm{c}}^{\mathrm{op}}(o, m) = \dot{\circlearrowleft}(o, m)$.

Let us note that there exists a strict category-like cocartesian category structure.

Let o, m be arbitrary. Then $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ is a strict category-like cocartesian category structure.

One can prove the following propositions:

(48) For every object a of $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ holds $a = o$.

(49) For all objects a, b of $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ holds $a = b$.

(50) For every morphism f of $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ holds $f = m$.

(51) For all morphisms f, g of $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ holds $f = g$.

(52) For all objects a, b of $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ and for every morphism f of $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ holds $f \in \hom(a,b)$.

(53) For all objects a, b of $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ every morphism of $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ is a morphism from a to b.

(54) For all objects a, b of $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ holds $\hom(a,b) \neq \emptyset$.

(55) Every object of $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ is an initial object.

(56) For every object c of $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ and for all morphisms i_1, i_2 of $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ holds c is a coproduct w.r.t. i_1 and i_2.

A category-like cocartesian category structure is cocartesian if:

(Def.25) the initial of it is an initial object and for all objects a, b of it holds dom (the 1st-coprojection of it)($\langle a, b\rangle$) $= a$ and dom (the 2nd-coprojection of it)($\langle a, b\rangle$) $= b$ and (the coproduct of it)($\langle a, b\rangle$) is a coproduct w.r.t. (the 1st-coprojection of it)($\langle a, b\rangle$) and (the 2nd-coprojection of it)($\langle a, b\rangle$).

One can prove the following proposition

(57) For arbitrary o, m holds $\dot{\circlearrowleft}^{\mathrm{op}}_{\mathrm{c}}(o,m)$ is cocartesian.

One can check that there exists a category-like cocartesian category structure which is strict and cocartesian.

A cocartesian category is a category-like cocartesian category structure.

We adopt the following rules: C is a cocartesian category and a, b, c, d, e, s are objects of C. Next we state two propositions:

(58) $\mathbf{0}_C$ is an initial object.

(59) For all morphisms f_1, f_2 from $\mathbf{0}_C$ to a holds $f_1 = f_2$.

Let us consider C, a. $!^a$ is a morphism from $\mathbf{0}_C$ to a.

We now state a number of propositions:

(60) $\hom(\mathbf{0}_C, a) \neq \emptyset$.

(61) $!^a = |^{\mathbf{0}_C} a$.

(62) $\mathrm{dom}(!^a) = \mathbf{0}_C$ and $\mathrm{cod}(!^a) = a$.

(63) $\hom(\mathbf{0}_C, a) = \{!^a\}$.

(64) $\mathrm{dom}\,\mathrm{in}_1(a+b) = a$ and $\mathrm{cod}\,\mathrm{in}_1(a+b) = a+b$.

(65) $\mathrm{dom}\,\mathrm{in}_2(a+b) = b$ and $\mathrm{cod}\,\mathrm{in}_2(a+b) = a+b$.

(66) $\hom(a, a+b) \neq \emptyset$ and $\hom(b, a+b) \neq \emptyset$.

(67) $a+b$ is a coproduct w.r.t. $\mathrm{in}_1(a+b)$ and $\mathrm{in}_2(a+b)$.

(68) C has finite coproduct.

(69) If $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$, then $\mathrm{in}_1(a+b)$ is coretraction and $\mathrm{in}_2(a+b)$ is coretraction.

Let us consider C, a, b. Then $\mathrm{in}_1(a+b)$ is a morphism from a to $a+b$. Then $\mathrm{in}_2(a+b)$ is a morphism from b to $a+b$. Let us consider C, a, b, c, and let f be a morphism from a to c, and let g be a morphism from b to c. Let us assume that $\hom(a,c) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$. The functor $\langle f,g\rangle$ yielding a morphism from $a+b$ to c is defined as follows:

(Def.26) $\langle f,g\rangle \cdot \mathrm{in}_1(a+b) = f$ and $\langle f,g\rangle \cdot \mathrm{in}_2(a+b) = g$.

Next we state several propositions:

(70) If $\hom(a,c) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$, then $\hom(a+b,c) \neq \emptyset$.

(71) $\langle \mathrm{in}_1(a+b), \mathrm{in}_2(a+b)\rangle = \mathrm{id}_{(a+b)}$.

(72) For every morphism f from a to c and for every morphism g from b to c and for every morphism h from c to d such that $\hom(a,c) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and $\hom(c,d) \neq \emptyset$ holds $\langle h\cdot f, h\cdot g\rangle = h \cdot \langle f,g\rangle$.

(73) For all morphisms f, k from a to c and for all morphisms g, h from b to c such that $\hom(a,c) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and $\langle f,g\rangle = \langle k,h\rangle$ holds $f=k$ and $g=h$.

(74) For every morphism f from a to c and for every morphism g from b to c such that $\hom(a,c) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and also f is epi or g is epi holds $\langle f,g\rangle$ is epi.

(75) a and $a+\mathbf{0}_C$ are isomorphic and a and $\mathbf{0}_C+a$ are isomorphic.

(76) $a+b$ and $b+a$ are isomorphic.

(77) $(a+b)+c$ and $a+(b+c)$ are isomorphic.

We now define two new functors. Let us consider C, a. The functor ∇_a yields a morphism from $a+a$ to a and is defined by:

(Def.27) $\nabla_a = \langle \mathrm{id}_a, \mathrm{id}_a\rangle$.

Let us consider C, a, b, c, d, and let f be a morphism from a to c, and let g be a morphism from b to d. The functor $f+g$ yielding a morphism from $a+b$ to $c+d$ is defined as follows:

(Def.28) $f+g = \langle \mathrm{in}_1(c+d)\cdot f, \mathrm{in}_2(c+d)\cdot g\rangle$.

The following propositions are true:

(78) If $\hom(a,c) \neq \emptyset$ and $\hom(b,d) \neq \emptyset$, then $\hom(a+b,c+d) \neq \emptyset$.

(79) $\mathrm{id}_a + \mathrm{id}_b = \mathrm{id}_{(a+b)}$.

(80) Let f be a morphism from a to c. Let h be a morphism from b to d. Then for every morphism g from c to e and for every morphism k from d to e such that $\hom(a,c) \neq \emptyset$ and $\hom(b,d) \neq \emptyset$ and $\hom(c,e) \neq \emptyset$ and $\hom(d,e) \neq \emptyset$ holds $\langle g,k\rangle \cdot (f+h) = \langle g\cdot f, k\cdot h\rangle$.

(81) For every morphism f from a to c and for every morphism g from b to c such that $\hom(a,c) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ holds $\nabla_c \cdot (f+g) = \langle f,g\rangle$.

(82) Let f be a morphism from a to c. Let h be a morphism from b to d. Then for every morphism g from c to e and for every morphism k from d to s such that $\hom(a, c) \neq \emptyset$ and $\hom(b, d) \neq \emptyset$ and $\hom(c, e) \neq \emptyset$ and $\hom(d, s) \neq \emptyset$ holds $(g + k) \cdot (f + h) = g \cdot f + k \cdot h$.

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