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Some Properties of Binary Relations

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Summary. The article contains some theorems on binary relations, which are used in papers [2], [3], [1], and other.

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The articles [5], [6], [7], and [4] provide the terminology and notation for this paper. We adopt the following rules: x, y are arbitrary, X, Y, Z, W are sets, and R, S, T are binary relations. We now state a number of propositions:

(1) If $X \cap Y = \emptyset$ and $x \in X \cup Y$, then $x \in X$ and $x \notin Y$ or $x \in Y$ and $x \notin X$.

(2) $(X \cup Y) \cup Z = X \cup Z \cup (Y \cup Z)$.

(3) $X \cup (X \cup Y) = X \cup Y$.

(4) If $X \subseteq Y \cap Z$, then $X \subseteq Y$ and $X \subseteq Z$.

(5) $\varnothing = \emptyset$.

(6) $\varnothing \setminus R = \varnothing$.

(7) $R \subseteq S$ if and only if $R \setminus S = \varnothing$.

(8) $R \cap S = \varnothing$ if and only if $R \setminus S = R$.

(9) $R \setminus R = \varnothing$.

(10) If $R \subseteq \varnothing$, then $R = \varnothing$.

(11) $\varnothing \cup R = R$ and $R \cup \varnothing = R$ and $\varnothing \cap R = \varnothing$ and $R \cap \varnothing = \varnothing$.

Let us consider X, Y. Then $[\!: X, Y :\!]$ is a binary relation.

Next we state several propositions:

(12) If $X \neq \emptyset$ and $Y \neq \emptyset$, then $\mathrm{dom}[\!: X, Y :\!] = X$ and $\mathrm{rng}[\!: X, Y :\!] = Y$.

(13) $\mathrm{dom}(R \cap [\!: X, Y :\!]) \subseteq X$ and $\mathrm{rng}(R \cap [\!: X, Y :\!]) \subseteq Y$.

(14) If $X \cap Y = \emptyset$, then $\mathrm{dom}(R \cap [\!: X, Y :\!]) \cap \mathrm{rng}(R \cap [\!: X, Y :\!]) = \emptyset$ and $\mathrm{dom}(R^{\smile} \cap [\!: X, Y :\!]) \cap \mathrm{rng}(R^{\smile} \cap [\!: X, Y :\!]) = \emptyset$.

(15) If $R \subseteq [\!: X, Y :\!]$, then $\mathrm{dom}\, R \subseteq X$ and $\mathrm{rng}\, R \subseteq Y$.

(16) If $R \subseteq [\!: X, Y :\!]$, then $R^{\smile} \subseteq [\!: Y, X :\!]$.

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(17) If $X \cap Y = \emptyset$, then $[\!: X, Y :\!] \cap [\!: Y, X :\!] = \emptyset$.

(18) $[\!: X, Y :\!]^{\smile} = [\!: Y, X :\!]$.

Next we state a number of propositions:

(19) $(R \cup S) \cdot T = R \cdot T \cup S \cdot T$ and $R \cdot (S \cup T) = R \cdot S \cup R \cdot T$.

(20) If $R \subseteq [\!: X, Y :\!]$ and $\langle x, y \rangle \in R$, then $x \in X$ and $y \in Y$.

(21) (i) If $X \cap Y = \emptyset$ and $R \subseteq [\!: X, Y :\!] \cup [\!: Y, X :\!]$ and $\langle x, y \rangle \in R$ and $x \in X$, then $x \notin Y$ and $y \notin X$ and $y \in Y$,

(ii) if $X \cap Y = \emptyset$ and $R \subseteq [\!: X, Y :\!] \cup [\!: Y, X :\!]$ and $\langle x, y \rangle \in R$ and $y \in Y$, then $y \notin X$ and $x \notin Y$ and $x \in X$,

(iii) if $X \cap Y = \emptyset$ and $R \subseteq [\!: X, Y :\!] \cup [\!: Y, X :\!]$ and $\langle x, y \rangle \in R$ and $x \in Y$, then $x \notin X$ and $y \notin Y$ and $y \in X$,

(iv) if $X \cap Y = \emptyset$ and $R \subseteq [\!: X, Y :\!] \cup [\!: Y, X :\!]$ and $\langle x, y \rangle \in R$ and $y \in X$, then $x \notin X$ and $y \notin Y$ and $x \in Y$.

(22) If $\operatorname{rng} R \cap \operatorname{dom} S = \emptyset$ or $\operatorname{dom} S \cap \operatorname{rng} R = \emptyset$, then $R \cdot S = \varnothing$.

(23) If $R \subseteq [\!: X, Y :\!]$ and $Z \subseteq X$, then $R \restriction Z = R \cap [\!: Z, Y :\!]$ but if $R \subseteq [\!: X, Y :\!]$ and $Z \subseteq Y$, then $Z \restriction R = R \cap [\!: X, Z :\!]$.

(24) If $R \subseteq [\!: X, Y :\!]$ and $X = Z \cup W$, then $R = R \restriction Z \cup R \restriction W$.

(25) If $X \cap Y = \emptyset$ and $R \subseteq [\!: X, Y :\!] \cup [\!: Y, X :\!]$, then $R \restriction X \subseteq [\!: X, Y :\!]$.

(26) If $R \subseteq S$, then $R^{\smile} \subseteq S^{\smile}$.

(27) $\triangle_X \subseteq [\!: X, X :\!]$.

(28) $\triangle_X \cdot \triangle_X = \triangle_X$.

(29) $\triangle_{\{x\}} = \{\langle x, x \rangle\}$.

(30) $\langle x, y \rangle \in \triangle_X$ if and only if $\langle y, x \rangle \in \triangle_X$.

(31) $\triangle_{X \cup Y} = \triangle_X \cup \triangle_Y$ and $\triangle_{X \cap Y} = \triangle_X \cap \triangle_Y$ and $\triangle_{X \setminus Y} = \triangle_X \setminus \triangle_Y$.

(32) If $X \subseteq Y$, then $\triangle_X \subseteq \triangle_Y$.

(33) $\triangle_{X \setminus Y} \setminus \triangle_X = \varnothing$.

(34) If $R \subseteq \triangle_{\operatorname{dom} R}$, then $R = \triangle_{\operatorname{dom} R}$.

(35) If $\triangle_X \subseteq R \cup R^{\smile}$, then $\triangle_X \subseteq R$ and $\triangle_X \subseteq R^{\smile}$.

(36) If $\triangle_X \subseteq R$, then $\triangle_X \subseteq R^{\smile}$.

(37) If $R \subseteq [\!: X, X :\!]$, then $R \setminus \triangle_{\operatorname{dom} R} = R \setminus \triangle_X$ and $R \setminus \triangle_{\operatorname{rng} R} = R \setminus \triangle_X$.

(38) If $\triangle_X \cdot (R \setminus \triangle_X) = \varnothing$, then $\operatorname{dom}(R \setminus \triangle_X) = \operatorname{dom} R \setminus \operatorname{dom}(\triangle_X)$ but if $(R \setminus \triangle_X) \cdot \triangle_X = \varnothing$, then $\operatorname{rng}(R \setminus \triangle_X) = \operatorname{rng} R \setminus \operatorname{rng}(\triangle_X)$.

(39) If $R \subseteq R \cdot R$ and $R \cdot (R \setminus \triangle_{\operatorname{rng} R}) = \varnothing$, then $\triangle_{\operatorname{rng} R} \subseteq R$ but if $R \subseteq R \cdot R$ and $(R \setminus \triangle_{\operatorname{dom} R}) \cdot R = \varnothing$, then $\triangle_{\operatorname{dom} R} \subseteq R$.

(40) (i) If $R \subseteq R \cdot R$ and $R \cdot (R \setminus \triangle_{\operatorname{rng} R}) = \varnothing$, then $R \cap \triangle_{\operatorname{rng} R} = \triangle_{\operatorname{rng} R}$,

(ii) if $R \subseteq R \cdot R$ and $(R \setminus \triangle_{\operatorname{dom} R}) \cdot R = \varnothing$, then $R \cap \triangle_{\operatorname{dom} R} = \triangle_{\operatorname{dom} R}$.

(41) If $R \cdot (R \setminus \triangle_X) = \varnothing$ and $\operatorname{rng} R \subseteq X$, then $R \cdot (R \setminus \triangle_{\operatorname{rng} R}) = \varnothing$ but if $(R \setminus \triangle_X) \cdot R = \varnothing$ and $\operatorname{dom} R \subseteq X$, then $(R \setminus \triangle_{\operatorname{dom} R}) \cdot R = \varnothing$.

Let us consider R. The functor $\mathrm{CL}(R)$ yielding a binary relation is defined as follows:

(Def.1) $\mathrm{CL}(R) = R \cap \triangle_{\operatorname{dom} R}$.

One can prove the following propositions:

(42) $\mathrm{CL}(R) \subseteq R$ and $\mathrm{CL}(R) \subseteq \triangle_{\mathrm{dom}\, R}$.

(43) If $\langle x, y \rangle \in \mathrm{CL}(R)$, then $x \in \mathrm{dom}\, \mathrm{CL}(R)$ and $x = y$.

(44) $\mathrm{dom}\, \mathrm{CL}(R) = \mathrm{rng}\, \mathrm{CL}(R)$.

(45) (i) $x \in \mathrm{dom}\, \mathrm{CL}(R)$ if and only if $x \in \mathrm{dom}\, R$ and $\langle x, x \rangle \in R$,
(ii) $x \in \mathrm{rng}\, \mathrm{CL}(R)$ if and only if $x \in \mathrm{dom}\, R$ and $\langle x, x \rangle \in R$,
(iii) $x \in \mathrm{rng}\, \mathrm{CL}(R)$ if and only if $x \in \mathrm{rng}\, R$ and $\langle x, x \rangle \in R$,
(iv) $x \in \mathrm{dom}\, \mathrm{CL}(R)$ if and only if $x \in \mathrm{rng}\, R$ and $\langle x, x \rangle \in R$.

(46) $\mathrm{CL}(R) = \triangle_{\mathrm{dom}\, \mathrm{CL}(R)}$.

(47) (i) If $R \cdot R = R$ and $R \cdot (R \setminus \mathrm{CL}(R)) = \varnothing$ and $\langle x, y \rangle \in R$ and $x \neq y$, then $x \in \mathrm{dom}\, R \setminus \mathrm{dom}\, \mathrm{CL}(R)$ and $y \in \mathrm{dom}\, \mathrm{CL}(R)$,
(ii) if $R \cdot R = R$ and $(R \setminus \mathrm{CL}(R)) \cdot R = \varnothing$ and $\langle x, y \rangle \in R$ and $x \neq y$, then $y \in \mathrm{rng}\, R \setminus \mathrm{dom}\, \mathrm{CL}(R)$ and $x \in \mathrm{dom}\, \mathrm{CL}(R)$.

(48) (i) If $R \cdot R = R$ and $R \cdot (R \setminus \triangle_{\mathrm{dom}\, R}) = \varnothing$ and $\langle x, y \rangle \in R$ and $x \neq y$, then $x \in \mathrm{dom}\, R \setminus \mathrm{dom}\, \mathrm{CL}(R)$ and $y \in \mathrm{dom}\, \mathrm{CL}(R)$,
(ii) if $R \cdot R = R$ and $(R \setminus \triangle_{\mathrm{dom}\, R}) \cdot R = \varnothing$ and $\langle x, y \rangle \in R$ and $x \neq y$, then $y \in \mathrm{rng}\, R \setminus \mathrm{dom}\, \mathrm{CL}(R)$ and $x \in \mathrm{dom}\, \mathrm{CL}(R)$.

(49) (i) If $R \cdot R = R$ and $R \cdot (R \setminus \triangle_{\mathrm{dom}\, R}) = \varnothing$, then $\mathrm{dom}\, \mathrm{CL}(R) = \mathrm{rng}\, R$ and $\mathrm{rng}\, \mathrm{CL}(R) = \mathrm{rng}\, R$,
(ii) if $R \cdot R = R$ and $(R \setminus \triangle_{\mathrm{dom}\, R}) \cdot R = \varnothing$, then $\mathrm{dom}\, \mathrm{CL}(R) = \mathrm{dom}\, R$ and $\mathrm{rng}\, \mathrm{CL}(R) = \mathrm{dom}\, R$.

(50) $\mathrm{dom}\, \mathrm{CL}(R) \subseteq \mathrm{dom}\, R$ and $\mathrm{rng}\, \mathrm{CL}(R) \subseteq \mathrm{rng}\, R$ and $\mathrm{rng}\, \mathrm{CL}(R) \subseteq \mathrm{dom}\, R$ and $\mathrm{dom}\, \mathrm{CL}(R) \subseteq \mathrm{rng}\, R$.

(51) $\triangle_{\mathrm{dom}\, \mathrm{CL}(R)} \subseteq \triangle_{\mathrm{dom}\, R}$ and $\triangle_{\mathrm{rng}\, \mathrm{CL}(R)} \subseteq \triangle_{\mathrm{dom}\, R}$.

(52) $\triangle_{\mathrm{dom}\, \mathrm{CL}(R)} \subseteq R$ and $\triangle_{\mathrm{rng}\, \mathrm{CL}(R)} \subseteq R$.

(53) If $\triangle_X \subseteq R$ and $\triangle_X \cdot (R \setminus \triangle_X) = \varnothing$, then $R \restriction X = \triangle_X$ but if $\triangle_X \subseteq R$ and $(R \setminus \triangle_X) \cdot \triangle_X = \varnothing$, then $X \restriction R = \triangle_X$.

(54) (i) If $\triangle_{\mathrm{dom}\, \mathrm{CL}(R)} \cdot (R \setminus \triangle_{\mathrm{dom}\, \mathrm{CL}(R)}) = \varnothing$, then $R \restriction \mathrm{dom}\, \mathrm{CL}(R) = \triangle_{\mathrm{dom}\, \mathrm{CL}(R)}$ and $R \restriction \mathrm{rng}\, \mathrm{CL}(R) = \triangle_{\mathrm{dom}\, \mathrm{CL}(R)}$,
(ii) if $(R \setminus \triangle_{\mathrm{rng}\, \mathrm{CL}(R)}) \cdot \triangle_{\mathrm{rng}\, \mathrm{CL}(R)} = \varnothing$, then $\mathrm{dom}\, \mathrm{CL}(R) \restriction R = \triangle_{\mathrm{dom}\, \mathrm{CL}(R)}$ and $\mathrm{rng}\, \mathrm{CL}(R) \restriction R = \triangle_{\mathrm{rng}\, \mathrm{CL}(R)}$.

(55) If $R \cdot (R \setminus \triangle_{\mathrm{dom}\, R}) = \varnothing$, then $\triangle_{\mathrm{dom}\, \mathrm{CL}(R)} \cdot (R \setminus \triangle_{\mathrm{dom}\, \mathrm{CL}(R)}) = \varnothing$ but if $(R \setminus \triangle_{\mathrm{dom}\, R}) \cdot R = \varnothing$, then $(R \setminus \triangle_{\mathrm{dom}\, \mathrm{CL}(R)}) \cdot \triangle_{\mathrm{dom}\, \mathrm{CL}(R)} = \varnothing$.

(56) (i) If $S \cdot R = S$ and $R \cdot (R \setminus \triangle_{\mathrm{dom}\, R}) = \varnothing$, then $S \cdot (R \setminus \triangle_{\mathrm{dom}\, R}) = \varnothing$,
(ii) if $R \cdot S = S$ and $(R \setminus \triangle_{\mathrm{dom}\, R}) \cdot R = \varnothing$, then $(R \setminus \triangle_{\mathrm{dom}\, R}) \cdot S = \varnothing$.

(57) If $S \cdot R = S$ and $R \cdot (R \setminus \triangle_{\mathrm{dom}\, R}) = \varnothing$, then $\mathrm{CL}(S) \subseteq \mathrm{CL}(R)$ but if $R \cdot S = S$ and $(R \setminus \triangle_{\mathrm{dom}\, R}) \cdot R = \varnothing$, then $\mathrm{CL}(S) \subseteq \mathrm{CL}(R)$.

(58) (i) If $S \cdot R = S$ and $R \cdot (R \setminus \triangle_{\mathrm{dom}\, R}) = \varnothing$ and $R \cdot S = R$ and $S \cdot (S \setminus \triangle_{\mathrm{dom}\, S}) = \varnothing$, then $\mathrm{CL}(S) = \mathrm{CL}(R)$,
(ii) if $R \cdot S = S$ and $(R \setminus \triangle_{\mathrm{dom}\, R}) \cdot R = \varnothing$ and $S \cdot R = R$ and $(S \setminus \triangle_{\mathrm{dom}\, S}) \cdot S = \varnothing$, then $\mathrm{CL}(S) = \mathrm{CL}(R)$.

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