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Reper Algebras

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Summary. We shall describe n-dimensional spaces with the reper operation [10, pages 72–79]. An inspiration to such approach comes from the monograph [12] and so-called Leibniz program. Let us recall that the Leibniz program is a program of algebraization of geometry using purely geometric notions. Leibniz formulated his program in opposition to algebraization method developed by Descartes. The Euclidean geometry in Szmielew's approach [12] is a theory of structures $\langle S;\, \|, \oplus, O\rangle$, where $\langle S;\, \|, \oplus, O\rangle$ is Desarguean midpoint plane and $O \subseteq S \times S \times S$ is the relation of equi-orthogonal basis. Points o, p, q are in relation O if they form an isosceles triangle with the right angle in vertex a. If we fix vertices a, p, then there exist exactly two points q, q' such that $O(apq)$, $O(apq')$. Moreover $q \oplus q' = a$. In accordance with the Leibniz program we replace the relation of equi-orthogonal basis by a binary operation $* : S \times S \to S$, called the reper operation. A standard model for the Euclidean geometry in the above sense is the oriented plane over the field of real numbers with the reper operations $*$ defined by the condition: $a * b = q$ iff the point q is the result of rotating of p about right angle around the center a.

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The terminology and notation used here are introduced in the following articles: [13], [5], [6], [3], [7], [2], [4], [1], [8], [11], and [9].

1. Substitutions in tuples

For simplicity we adopt the following rules: n, i, j, k, l are natural numbers, D is a non-empty set, c, d are elements of D, and p, q, r are finite sequences of elements of D. The following propositions are true:

(1) If $\operatorname{len} p = j + 1 + k$, then there exist q, r, c such that $\operatorname{len} q = j$ and $\operatorname{len} r = k$ and $p = q \frown \langle c\rangle \frown r$.

(2) If $i \in \operatorname{Seg} n$, then there exist j, k such that $n = j + 1 + k$ and $i = j + 1$.

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(3) Suppose $p = q \frown \langle c \rangle \frown r$ and $i = \operatorname{len} q + 1$. Then for every l such that $1 \leq l$ and $l \leq \operatorname{len} q$ holds $p(l) = q(l)$ and $p(i) = c$ and for every l such that $i + 1 \leq l$ and $l \leq \operatorname{len} p$ holds $p(l) = r(l - i)$.

(4) $l \leq j$ or $l = j + 1$ or $j + 2 \leq l$.

(5) If $l \in \operatorname{Seg} n \setminus \{i\}$ and $i = j + 1$, then $1 \leq l$ and $l \leq j$ or $i + 1 \leq l$ and $l \leq n$.

Let us consider n, i, D, d, and let p be an element of D^{n+1}. Let us assume that $i \in \operatorname{Seg}(n + 1)$. The functor $p(i/d)$ yielding an element of D^{n+1} is defined as follows:

(Def.1) $p(i/d)(i) = d$ and for every l such that $l \in \operatorname{Seg} \operatorname{len} p \setminus \{i\}$ holds $p(i/d)(l) = p(l)$.

2. Reper Algebra Structure and its Properties

Let us consider n. We consider structures of reper algebra over n which are extension of a midpoint algebra structure and are systems

⟨a carrier, a midpoint operation, a reper⟩,

where the carrier is a non-empty set, the midpoint operation is a binary operation on the carrier, and the reper is a function from (the carrier)^n into the carrier. Let us observe that there exists a structure of reper algebra over $n + 2$ which is midpoint algebra-like.

We adopt the following rules: R_1 will denote a midpoint algebra-like structure of reper algebra over $n + 2$ and a, b, d, p_1, p'_1 will denote points of R_1. We now define two new modes. Let us consider i, D. A tuple of i and D is an element of D^i.

Let us consider n, R_1, i. A tuple of i and R_1 is a tuple of i and the carrier of R_1.

In the sequel p, q will denote tuples of $n + 1$ and R_1. Let us consider n, R_1, a. Then $\langle a \rangle$ is a tuple of 1 and R_1. Let us consider n, R_1, i, j, and let p be a tuple of i and R_1, and let q be a tuple of j and R_1. Then $p \frown q$ is a tuple of $i + j$ and R_1.

We now state the proposition

(6) $\langle a \rangle \frown p$ is a tuple of $n + 2$ and R_1.

We now define two new functors. Let us consider n, R_1, a, p. The functor $*(a, p)$ yielding a point of R_1 is defined as follows:

(Def.2) $*(a, p) = (\text{the reper of } R_1)(\langle a \rangle \frown p)$.

Let us consider n, i, R_1, d, p. The functor $p_{\restriction i \dot{\to} d}$ yields a tuple of $n + 1$ and R_1 and is defined as follows:

(Def.3) for every D and for every element p' of D^{n+1} and for every element d' of D such that $D =$ the carrier of R_1 and $p' = p$ and $d' = d$ holds $p_{\restriction i \dot{\to} d} = p'(i/d')$.

We now state the proposition

(7) If $i \in \operatorname{Seg}(n+1)$, then $p_{\restriction i \dot{\to} d}(i) = d$ and for every l such that $l \in \operatorname{Seg} \operatorname{len} p \setminus \{i\}$ holds $p_{\restriction i \dot{\to} d}(l) = p(l)$.

Let us consider n. A natural number is said to be a natural number of n if:

(Def.4) $1 \leq$ it and it $\leq n+1$.

In the sequel m is a natural number of n. We now state several propositions:

(8) i is a natural number of n if and only if $i \in \operatorname{Seg}(n+1)$.

(9) $1 \leq i+1$.

(10) If $i \leq n$, then $i+1$ is a natural number of n.

(11) If for every m holds $p(m) = q(m)$, then $p = q$.

(12) For every natural number l of n such that $l = i$ holds $p_{\restriction i \dot{\to} d}(l) = d$ and for all natural numbers l, i of n such that $l \neq i$ holds $p_{\restriction i \dot{\to} d}(l) = p(l)$.

We now define three new predicates. Let us consider n, D, and let p be an element of D^{n+1}, and let us consider m. Then $p(m)$ is an element of D. Let us consider n, R_1. We say that R_1 is invariance if and only if:

(Def.5) for all a, b, p, q such that for every m holds $a \oplus q(m) = b \oplus p(m)$ holds $a \oplus *(b, q) = b \oplus *(a, p)$.

Let us consider n, i, R_1. We say that R_1 has property of zero in i if and only if:

(Def.6) for all a, p holds $*(a, p_{\restriction i \dot{\to} a}) = a$.

We say that R_1 is semi additive in i if and only if:

(Def.7) for all a, p_1, p such that $p(i) = p_1$ holds $*(a, p_{\restriction i \dot{\to} a \oplus p_1}) = a \oplus *(a, p)$.

The following proposition is true

(13) If R_1 is semi additive in m, then for all a, d, p, q such that $q = p_{\restriction m \dot{\to} d}$ holds $*(a, p_{\restriction m \dot{\to} a \oplus d}) = a \oplus *(a, q)$.

We now define two new predicates. Let us consider n, i, R_1. We say that R_1 is additive in i if and only if:

(Def.8) for all a, p_1, p_1', p such that $p(i) = p_1$ holds $*(a, p_{\restriction i \dot{\to} p_1 \oplus p_1'}) = *(a, p) \oplus *(a, p_{\restriction i \dot{\to} p_1'})$.

We say that R_1 is alternative in i if and only if:

(Def.9) for all a, p, p_1 such that $p(i) = p_1$ holds $*(a, p_{\restriction i+1 \dot{\to} p_1}) = a$.

In the sequel W is an atlas of R_1 and v is a vector of W. Let us consider n, R_1, W, i. A tuple of i and W is a tuple of i and the carrier of the algebra of W.

In the sequel x, y are tuples of $n+1$ and W. Let us consider n, R_1, W, x, i, v. The functor $x_{\restriction i \dot{\to} v}$ yields a tuple of $n+1$ and W and is defined by:

(Def.10) for every D and for every element x' of D^{n+1} and for every element v' of D such that $D =$ the carrier of the algebra of W and $x' = x$ and $v' = v$ holds $x_{\restriction i \dot{\to} v} = x'(i/v')$.

Next we state three propositions:

(14) If $i \in \operatorname{Seg}(n+1)$, then $x_{\restriction i \dot{\to} v}(i) = v$ and for every l such that $l \in \operatorname{Seg} \operatorname{len} x \setminus \{i\}$ holds $x_{\restriction i \dot{\to} v}(l) = x(l)$.

(15) For every natural number l of n such that $l = i$ holds $x_{\restriction i \dot{\rightarrow} v}(l) = v$ and for all natural numbers l, i of n such that $l \neq i$ holds $x_{\restriction i \dot{\rightarrow} v}(l) = x(l)$.

(16) If for every m holds $x(m) = y(m)$, then $x = y$.

The scheme *SeqLambdaD'* concerns a natural number $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

there exists a finite sequence z of elements of $\mathcal{B}$ such that $\operatorname{len} z = \mathcal{A} + 1$ and for every natural number j of $\mathcal{A}$ holds $z(j) = \mathcal{F}(j)$

for all values of the parameters.

We now define two new functors. Let us consider n, R_1, W, a, x. The functor $(a, x).W$ yielding a tuple of $n+1$ and R_1 is defined as follows:

(Def.11) $((a, x).W)(m) = (a, x(m)).W$.

Let us consider n, R_1, W, a, p. The functor $W(a, p)$ yielding a tuple of $n+1$ and W is defined by:

(Def.12) $W(a, p)(m) = W(a, p(m))$.

The following three propositions are true:

(17) $W(a, p) = x$ if and only if $(a, x).W = p$.

(18) $W(a, (a, x).W) = x$.

(19) $(a, W(a, p)).W = p$.

Let us consider n, R_1, W, a, x. The functor $\Phi(a, x)$ yields a vector of W and is defined by:

(Def.13) $\Phi(a, x) = W(a, *(a, (a, x).W))$.

One can prove the following propositions:

(20) If $W(a, p) = x$ and $W(a, b) = v$, then $*(a, p) = b$ if and only if $\Phi(a, x) = v$.

(21) R_1 is invariance if and only if for all a, b, x holds $\Phi(a, x) = \Phi(b, x)$.

(22) $1 \in \operatorname{Seg}(n+1)$.

(23) 1 is an element of $\operatorname{Seg}(n+1)$.

(24) 1 is a natural number of n.

3. Reper Algebra and its Atlas

Let us consider n. A midpoint algebra-like structure of reper algebra over $n+2$ is called a reper algebra of n if:

(Def.14) it is invariance.

For simplicity we adopt the following convention: R_1 will be a reper algebra of n, a, b will be points of R_1, p will be a tuple of $n+1$ and R_1, W will be an atlas of R_1, v will be a vector of W, and x will be a tuple of $n+1$ and W. Next we state the proposition

(25) $\Phi(a, x) = \Phi(b, x)$.

Let us consider n, R_1, W, x. The functor $\Phi(x)$ yields a vector of W and is defined by:

(Def.15) for every a holds $\Phi(x) = \Phi(a, x)$.

We now state a number of propositions:

(26) If $W(a, p) = x$ and $W(a, b) = v$ and $\Phi(x) = v$, then $*(a, p) = b$.

(27) If $(a, x).W = p$ and $(a, v).W = b$ and $*(a, p) = b$, then $\Phi(x) = v$.

(28) If $W(a, p) = x$ and $W(a, b) = v$, then $W(a, p_{\restriction m \dot{\rightarrow} b}) = x_{\restriction m \dot{\rightarrow} v}$.

(29) If $(a, x).W = p$ and $(a, v).W = b$, then $(a, x_{\restriction m \dot{\rightarrow} v}).W = p_{\restriction m \dot{\rightarrow} b}$.

(30) R_1 has property of zero in m if and only if for every x holds $\Phi((x_{\restriction m \dot{\rightarrow} 0_W})) = 0_W$.

(31) R_1 is semi additive in m if and only if for every x holds $\Phi((x_{\restriction m \dot{\rightarrow} 2x(m)})) = 2\,\Phi(x)$.

(32) If R_1 has property of zero in m and R_1 is additive in m, then R_1 is semi additive in m.

(33) If R_1 has property of zero in m, then R_1 is additive in m if and only if for all x, v holds $\Phi((x_{\restriction m \dot{\rightarrow} x(m)+v})) = \Phi(x) + \Phi((x_{\restriction m \dot{\rightarrow} v}))$.

(34) If $W(a, p) = x$ and $m \leq n$, then $W(a, p_{\restriction m+1 \dot{\rightarrow} p(m)}) = x_{\restriction m+1 \dot{\rightarrow} x(m)}$.

(35) If $(a, x).W = p$ and $m \leq n$, then $(a, x_{\restriction m+1 \dot{\rightarrow} x(m)}).W = p_{\restriction m+1 \dot{\rightarrow} p(m)}$.

(36) If $m \leq n$, then R_1 is alternative in m if and only if for every x holds $\Phi((x_{\restriction m+1 \dot{\rightarrow} x(m)})) = 0_W$.

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