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Properties of Caratheodor's Measure

Józef Białas
University of Łódź

Summary. The paper contains definitions and basic properties of Caratheodory measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ - by [14]. The article includes the text being a continuation of the paper [3]. Caratheodory theorem and some theorems concerning basic properties of Caratheodory measure are proved. The work is the sixth part of the series of articles concerning the Lebesgue measure theory.

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The terminology and notation used in this paper have been introduced in the following papers: [16], [15], [10], [11], [8], [9], [1], [13], [2], [12], [4], [5], [7], [6], [3], and [17]. One can prove the following propositions:

(1) For all *Real numbers* x, y, z such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq y$ and $0_{\overline{\mathbb{R}}} \leq z$ holds $(x + y) + z = x + (y + z)$.

(2) For all *Real numbers* x, y, z such that $x \neq -\infty$ and $x \neq +\infty$ holds $y + x \leq z$ if and only if $y \leq z - x$.

(3) For all *Real numbers* x, y such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq y$ holds $x + y = y + x$.

(4) For every set X and for every σ-field S of subsets of X and for every function F from $\mathbb{N}$ into S and for every element A of S and for every function G from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $G(n) = A \cap F(n)$ holds $\bigcup \operatorname{rng} G = A \cap \bigcup \operatorname{rng} F$.

(5) Let X be a set. Let S be a σ-field of subsets of X. Let F be a function from $\mathbb{N}$ into S. Let G be a function from $\mathbb{N}$ into S. Suppose $G(0) = F(0)$ and for every element n of $\mathbb{N}$ holds $G(n + 1) = F(n + 1) \cup G(n)$. Then for every function H from $\mathbb{N}$ into S such that $H(0) = F(0)$ and for every element n of $\mathbb{N}$ holds $H(n+1) = F(n+1) \setminus G(n)$ holds $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} H$.

(6) For every set X holds 2^X is a σ-field of subsets of X.

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Let X be a set, and let F be a function from $\mathbb{N}$ into 2^X. Then $\operatorname{rng} F$ is a non-empty family of subsets of X. Let A be a non-empty family of subsets of X. Then $\bigcup A$ is an element of 2^X. Let F be a function from 2^X into $\overline{\mathbb{R}}$. We say that F is non-negative if and only if:

(Def.1) for every element A of 2^X holds $0_{\overline{\mathbb{R}}} \leq F(A)$.

Let F be a function from $\mathbb{N}$ into 2^X, and let M be a function from 2^X into $\overline{\mathbb{R}}$. Then $M \cdot F$ is a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$.

One can prove the following propositions:

(7) For every set X and for every *Real numbers* a, b there exists a function M from 2^X into $\overline{\mathbb{R}}$ such that for every element A of 2^X holds if $A = \emptyset$, then $M(A) = a$ but if $A \neq \emptyset$, then $M(A) = b$.

(8) For every set X there exists a function M from 2^X into $\overline{\mathbb{R}}$ such that for every element A of 2^X holds $M(A) = 0_{\overline{\mathbb{R}}}$.

(9) For every set X and for every function F from $\mathbb{N}$ into 2^X and for every function M from 2^X into $\overline{\mathbb{R}}$ such that M is non-negative holds $M \cdot F$ is non-negative.

(10) For every set X and for every function F from $\mathbb{N}$ into 2^X and for every function M from 2^X into $\overline{\mathbb{R}}$ and for every natural number n holds $(M \cdot F)(n) = M(F(n))$.

(11) Let X be a set. Then there exists a function M from 2^X into $\overline{\mathbb{R}}$ such that M is non-negative and $M(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for all elements A, B of 2^X such that $A \subseteq B$ holds $M(A) \leq M(B)$ and for every function F from $\mathbb{N}$ into 2^X holds $M(\bigcup \operatorname{rng} F) \leq \sum(M \cdot F)$.

We now define two new constructions. Let X be a set. A function from 2^X into $\overline{\mathbb{R}}$ is said to be a Caratheodor's measure on X if:

(Def.2) it is non-negative and $\mathrm{it}(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for all elements A, B of 2^X such that $A \subseteq B$ holds $\mathrm{it}(A) \leq \mathrm{it}(B)$ and for every function F from $\mathbb{N}$ into 2^X holds $\mathrm{it}(\bigcup \operatorname{rng} F) \leq \sum(\mathrm{it} \cdot F)$.

Let C be a Caratheodor's measure on X. The functor σ-Field(C) yielding a non-empty family of subsets of X is defined by:

(Def.3) for every element A of 2^X holds $A \in \sigma$-Field(C) if and only if for all elements W, Z of 2^X such that $W \subseteq A$ and $Z \subseteq X \setminus A$ holds $C(W) + C(Z) \leq C(W \cup Z)$.

The following propositions are true:

(12) For every set X and for every Caratheodor's measure C on X and for all elements W, Z of 2^X holds $C(W \cup Z) \leq C(W) + C(Z)$.

(13) For every set X and for every Caratheodor's measure C on X and for all elements W, Z of 2^X holds $C(Z) + C(W) = C(W) + C(Z)$.

(14) For every set X and for every Caratheodor's measure C on X and for every element A of 2^X holds $A \in \sigma$-Field(C) if and only if for all elements W, Z of 2^X such that $W \subseteq A$ and $Z \subseteq X \setminus A$ holds $C(W) + C(Z) = C(W \cup Z)$.

(15) For every set X and for every Caratheodor's measure C on X and for all elements W, Z of 2^X such that $W \in \sigma\text{-Field}(C)$ and $Z \in \sigma\text{-Field}(C)$ and $Z \cap W = \emptyset$ holds $C(W \cup Z) = C(W) + C(Z)$.

(16) For every set X and for every Caratheodor's measure C on X and for every set A such that $A \in \sigma\text{-Field}(C)$ holds $X \setminus A \in \sigma\text{-Field}(C)$.

(17) For every set X and for every Caratheodor's measure C on X and for all sets A, B such that $A \in \sigma\text{-Field}(C)$ and $B \in \sigma\text{-Field}(C)$ holds $A \cup B \in \sigma\text{-Field}(C)$.

(18) For every set X and for every Caratheodor's measure C on X and for all sets A, B such that $A \in \sigma\text{-Field}(C)$ and $B \in \sigma\text{-Field}(C)$ holds $A \cap B \in \sigma\text{-Field}(C)$.

(19) For every set X and for every Caratheodor's measure C on X and for all sets A, B such that $A \in \sigma\text{-Field}(C)$ and $B \in \sigma\text{-Field}(C)$ holds $A \setminus B \in \sigma\text{-Field}(C)$.

(20) For every set X and for every σ-field S of subsets of X and for every function N from $\mathbb{N}$ into S and for every element A of S there exists a function F from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $F(n) = A \cap N(n)$.

(21) For every set X and for every Caratheodor's measure C on X holds $\sigma\text{-Field}(C)$ is a σ-field of subsets of X.

Let X be a set, and let C be a Caratheodor's measure on X. Then $\sigma\text{-Field}(C)$ is a σ-field of subsets of X. Let S be a σ-field of subsets of X, and let A be a subfamily of S. Then $\bigcup A$ is an element of S. The functor $\sigma\text{-Meas}(C)$ yields a function from $\sigma\text{-Field}(C)$ into $\overline{\mathbb{R}}$ and is defined by:

(Def.4) for every element A of 2^X such that $A \in \sigma\text{-Field}(C)$ holds $(\sigma\text{-Meas}(C))(A) = C(A)$.

One can prove the following proposition

(22) For every set X and for every Caratheodor's measure C on X holds $\sigma\text{-Meas}(C)$ is a measure on $\sigma\text{-Field}(C)$.

Let X be a set, and let C be a Caratheodor's measure on X, and let A be an element of $\sigma\text{-Field}(C)$. Then $C(A)$ is a *Real number*.

One can prove the following proposition

(23) For every set X and for every Caratheodor's measure C on X holds $\sigma\text{-Meas}(C)$ is a σ-measure on $\sigma\text{-Field}(C)$.

Let X be a set, and let C be a Caratheodor's measure on X. Then $\sigma\text{-Meas}(C)$ is a σ-measure on $\sigma\text{-Field}(C)$.

The following propositions are true:

(24) For every set X and for every Caratheodor's measure C on X and for every element A of 2^X such that $C(A) = 0_{\overline{\mathbb{R}}}$ holds $A \in \sigma\text{-Field}(C)$.

(25) For every set X and for every Caratheodor's measure C on X holds $\sigma\text{-Meas}(C)$ is complete on $\sigma\text{-Field}(C)$.

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