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Completeness of the σ-Additive Measure. Measure Theory

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Summary. Definitions and basic properties of a σ-additive, non-negative measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ - by [13]. The article includs the text being a continuation of the paper [5]. Some theorems concerning basic properties of a σ-additive measure and completeness of the measure are proved.

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The papers [15], [14], [9], [10], [7], [8], [1], [12], [2], [11], [3], [4], [6], and [5] provide the terminology and notation for this paper. One can prove the following four propositions:

(1) For every *Real number* x such that $-\infty < x$ and $x < +\infty$ holds x is a real number.

(2) For every *Real number* x such that $x \neq -\infty$ and $x \neq +\infty$ holds x is a real number.

(3) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F_1 is non-negative and F_2 is non-negative holds if for every natural number n holds $(\mathrm{Ser}\, F_1)(n) \leq (\mathrm{Ser}\, F_2)(n)$, then $\sum F_1 \leq \sum F_2$.

(4) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F_1 is non-negative and F_2 is non-negative holds if for every natural number n holds $(\mathrm{Ser}\, F_1)(n) = (\mathrm{Ser}\, F_2)(n)$, then $\sum F_1 = \sum F_2$.

Let X be a set, and let S be a σ-field of subsets of X. A denumerable family of subsets of X is called a subfamily of S if:

(Def.1) it $\subseteq S$.

Let X be a set, and let S be a σ-field of subsets of X, and let F be a function from $\mathbb{N}$ into S. Then $\mathrm{rng}\, F$ is a subfamily of S.

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Let X be a set, and let S be a σ-field of subsets of X, and let A be a subfamily of S. Then $\bigcup A$ is an element of S.

Let X be a set, and let S be a σ-field of subsets of X, and let A be a subfamily of S. Then $\bigcap A$ is an element of S.

One can prove the following propositions:

(5) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into S and for every element A of S such that $\bigcap \operatorname{rng} F \subseteq A$ and for every element n of $\mathbb{N}$ holds $A \subseteq F(n)$ holds $M(A) = M(\bigcap \operatorname{rng} F)$.

(6) Let X be a set. Let S be a σ-field of subsets of X. Let G be a function from $\mathbb{N}$ into S. Then for every function F from $\mathbb{N}$ into S such that $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$ holds $\bigcup \operatorname{rng} G = F(0) \setminus \bigcap \operatorname{rng} F$.

(7) Let X be a set. Let S be a σ-field of subsets of X. Let G be a function from $\mathbb{N}$ into S. Then for every function F from $\mathbb{N}$ into S such that $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$ holds $\bigcap \operatorname{rng} F = F(0) \setminus \bigcup \operatorname{rng} G$.

(8) Let X be a set. Let S be a σ-field of subsets of X. Let M be a σ-measure on S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $M(\bigcap \operatorname{rng} F) = M(F(0)) - M(\bigcup \operatorname{rng} G)$.

(9) Let X be a set. Let S be a σ-field of subsets of X. Let M be a σ-measure on S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $M(\bigcup \operatorname{rng} G) = M(F(0)) - M(\bigcap \operatorname{rng} F)$.

(10) Let X be a set. Let S be a σ-field of subsets of X. Let M be a σ-measure on S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $M(\bigcap \operatorname{rng} F) = M(F(0)) - \sup \operatorname{rng}(M \cdot G)$.

(11) Let X be a set. Let S be a σ-field of subsets of X. Let M be a σ-measure on S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $\sup \operatorname{rng}(M \cdot G)$ is a real number and $M(F(0))$ is a real number and $\inf \operatorname{rng}(M \cdot F)$ is a real number.

(12) Let X be a set. Let S be a σ-field of subsets of X. Let M be a σ-measure on S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $\sup \operatorname{rng}(M \cdot G) = M(F(0)) - \inf \operatorname{rng}(M \cdot F)$.

(13) Let X be a set. Let S be a σ-field of subsets of X. Let M be a σ-measure on S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $\inf \operatorname{rng}(M \cdot F) = M(F(0)) - \sup \operatorname{rng}(M \cdot G)$.

(14) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $F(n+1) \subseteq F(n)$ and $M(F(0)) < +\infty$ holds $M(\bigcap \operatorname{rng} F) = \inf \operatorname{rng}(M \cdot F)$.

(15) For every set X and for every σ-field S of subsets of X and for every measure M on S and for every family T of measureable sets of S and for every sequence F of separated subsets of S such that $T = \operatorname{rng} F$ holds $\sum(M \cdot F) \leq M(\bigcup T)$.

(16) For every set X and for every σ-field S of subsets of X and for every measure M on S and for every sequence F of separated subsets of S holds $\sum(M \cdot F) \leq M(\bigcup \operatorname{rng} F)$.

(17) For every set X and for every σ-field S of subsets of X and for every measure M on S such that for every sequence F of separated subsets of S holds $M(\bigcup \operatorname{rng} F) \leq \sum(M \cdot F)$ holds M is a σ-measure on S.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S. We say that M is complete on S if and only if:

(Def.2) for every subset A of X and for every set B such that $B \in S$ holds if $A \subseteq B$ and $M(B) = 0_{\overline{\mathbb{R}}}$, then $A \in S$.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S. A subset of X is called a set with measure zero w.r.t. M if:

(Def.3) there exists a set B such that $B \in S$ and it $\subseteq B$ and $M(B) = 0_{\overline{\mathbb{R}}}$.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor $\mathrm{COM}(S, M)$ yielding a non-empty family of subsets of X is defined as follows:

(Def.4) for an arbitrary A holds $A \in \mathrm{COM}(S, M)$ if and only if there exists a set B such that $B \in S$ and there exists a set C with measure zero w.r.t. M such that $A = B \cup C$.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S, and let A be an element of $\mathrm{COM}(S, M)$. The functor $\mathrm{MeasPart}A$ yields a non-empty family of subsets of X and is defined as follows:

(Def.5) for an arbitrary B holds $B \in \mathrm{MeasPart}A$ if and only if $B \in S$ and $B \subseteq A$ and $A \setminus B$ is a set with measure zero w.r.t. M.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S, and let F be a function from $\mathbb{N}$ into $\mathrm{COM}(S, M)$, and let n be a natural number. Then $F(n)$ is an element of $\mathrm{COM}(S, M)$.

We now state four propositions:

(18) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into $\mathrm{COM}(S, M)$ there exists a function G from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $G(n) \in \mathrm{MeasPart}F(n)$.

(19) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into $\mathrm{COM}(S, M)$ and for every function G from $\mathbb{N}$ into S there exists a function H from $\mathbb{N}$ into 2^X such that for every element n of $\mathbb{N}$ holds $H(n) = F(n) \setminus G(n)$.

(20) Let X be a set. Then for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into 2^X such that for every element n of $\mathbb{N}$ holds $F(n)$ is a set with measure zero w.r.t. M there exists a function G from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $F(n) \subseteq G(n)$ and $M(G(n)) = 0_{\overline{\mathbb{R}}}$.

(21) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every non-empty family D of subsets of X such that for an arbitrary A holds $A \in D$ if and only if there exists a set B such that $B \in S$ and there exists a set C with measure zero w.r.t. M such that $A = B \cup C$ holds D is a σ-field of subsets of X.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S. Then $\mathrm{COM}(S, M)$ is a σ-field of subsets of X.

Next we state the proposition

(22) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for all sets B_1, B_2 such that $B_1 \in S$ and $B_2 \in S$ and for all sets C_1, C_2 with measure zero w.r.t. M such that $B_1 \cup C_1 = B_2 \cup C_2$ holds $M(B_1) = M(B_2)$.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor $\mathrm{COM}(M)$ yields a σ-measure on $\mathrm{COM}(S, M)$ and is defined by:

(Def.6) for every set B such that $B \in S$ and for every set C with measure zero w.r.t. M holds $(\mathrm{COM}(M))(B \cup C) = M(B)$.

The following proposition is true

(23) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S holds $\mathrm{COM}(M)$ is complete on $\mathrm{COM}(S, M)$.

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