FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Free Modules

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Summary. We define free modules and prove that every left module over Skew-Field is free.

MML Identifier: `MOD_3`.

The papers [20], [5], [3], [2], [4], [19], [16], [14], [15], [1], [18], [6], [7], [8], [12], [11], [9], [10], [13], and [17] provide the terminology and notation for this paper. One can prove the following propositions:

(1) For every ring R and for every scalar a of R such that $-a = 0_R$ holds $a = 0_R$.

(2) For every integral domain R holds $0_R \neq -1_R$.

For simplicity we follow the rules: x is arbitrary, R is an associative ring, V is a left module over R, L, L_1, L_2 are linear combinations of V, a is a scalar of R, v, w are vectors of V, F is a finite sequence of elements of the carrier of the carrier of V, and C is a finite subset of V. We now state several propositions:

(3) If $-v = w$, then $v = -w$.

(4) $\sum(\mathbf{0}_{\mathrm{LC}_V}) = \Theta_V$.

(5) $L_1 + L_2 = L_2 + L_1$.

(6) If $\mathrm{support}\, L \subseteq C$, then there exists F such that F is one-to-one and $\mathrm{rng}\, F = C$ and $\sum L = \sum(LF)$.

(7) $\sum(a \cdot L) = a \cdot \sum L$.

(8) $\sum(-L) = -\sum L$.

(9) $\sum(L_1 - L_2) = \sum L_1 - \sum L_2$.

(10) $L + \mathbf{0}_{\mathrm{LC}_V} = L$ and $\mathbf{0}_{\mathrm{LC}_V} + L = L$.

In the sequel W denotes a submodule of V, A, B denote subsets of V, and l denotes a linear combination of A. Let us consider R, V, A. The functor $\mathrm{Lin}(A)$ yielding a submodule of V is defined as follows:

ISSN 0777–4028

(Def.1) the carrier of the carrier of $\mathrm{Lin}(A) = \{\sum l\}$.

The following propositions are true:

(11) $x \in \mathrm{Lin}(A)$ if and only if there exists l such that $x = \sum l$.

(12) If $x \in A$, then $x \in \mathrm{Lin}(A)$.

(13) $\mathrm{Lin}(\emptyset_{\text{the carrier of the carrier of } V}) = \mathbf{0}_V$.

(14) If $\mathrm{Lin}(A) = \mathbf{0}_V$, then $A = \emptyset$ or $A = \{\Theta_V\}$.

(15) If $0_R \neq 1_R$ and $A =$ the carrier of the carrier of W, then $\mathrm{Lin}(A) = W$.

(16) If $0_R \neq 1_R$ and $A =$ the carrier of the carrier of V, then $\mathrm{Lin}(A) = V$.

(17) If $A \subseteq B$, then $\mathrm{Lin}(A)$ is a submodule of $\mathrm{Lin}(B)$.

(18) If $\mathrm{Lin}(A) = V$ and $A \subseteq B$, then $\mathrm{Lin}(B) = V$.

(19) $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(20) $\mathrm{Lin}(A \cap B)$ is a submodule of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

Let us consider R, V. A subset of V is base if:

(Def.2) it is linearly independent and $\mathrm{Lin}(\text{it}) = V$.

Let us consider R. A left module over R is free if:

(Def.3) there exists a subset B of it such that B is base.

We now state the proposition

(21) $\mathbf{0}_V$ is free.

Let us consider R. A left module over R is called a free left R-module if:

(Def.4) it is free.

For simplicity we adopt the following convention: R will denote a skew field, a, b will denote scalars of R, V will denote a left module over R, v, v_1, v_2 will denote vectors of V, and A, B will denote subsets of V. The following propositions are true:

(22) $0_R \neq -1_R$.

(23) $\{v\}$ is linearly independent if and only if $v \neq \Theta_V$.

(24) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if $v_2 \neq \Theta_V$ and for every a holds $v_1 \neq a \cdot v_2$.

(25) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if for all a, b such that $a \cdot v_1 + b \cdot v_2 = \Theta_V$ holds $a = 0_R$ and $b = 0_R$.

(26) If A is linearly independent, then there exists B such that $A \subseteq B$ and B is base.

(27) If $\mathrm{Lin}(A) = V$, then there exists B such that $B \subseteq A$ and B is base.

(28) V is free.

Let us consider R, V. A subset of V is called a basis of V if:

(Def.5) it is base.

In the sequel I is a basis of V. The following two propositions are true:

(29) If A is linearly independent, then there exists I such that $A \subseteq I$.

(30) If $\mathrm{Lin}(A) = V$, then there exists I such that $I \subseteq A$.

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Received October 18, 1991