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Introduction to Banach and Hilbert Spaces - Part I

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Summary. Basing on the notion of real linear space (see [15]) we introduce real unitary space. At first, we define the scalar product of two vectors and examine some of its properties. On the basis of this notion we introduce the norm and the distance in real unitary space and study the properties of these concepts. Next, proceeding from the definition of the sequence in real unitary space and basic operations on sequences we prove several theorems which will be used in our further considerations.

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The terminology and notation used here are introduced in the following articles: [5], [12], [16], [3], [4], [1], [6], [2], [17], [10], [11], [9], [15], [14], [13], [8], and [7].
We consider unitary space structures which are systems

$\langle$vectors, a scalar product$\rangle$,

where the vectors constitute a real linear space and the scalar product is a function from [: the vectors of the vectors, the vectors of the vectors :] into $\mathbb{R}$.

In the sequel X will denote a unitary space structure and a, b will denote real numbers. Let us consider X. A point of X is an element of the vectors of the vectors of X.

In the sequel x, y will denote points of X. Let us consider X, x, y. The functor $(x|y)$ yielding a real number is defined as follows:

(Def.1) $(x|y) = (\text{the scalar product of } X)(\langle x, y\rangle)$.

A unitary space structure is said to be a real unitary space if it satisfies the condition (Def.2).

(Def.2) Let x, y, z be points of it. Given a. Then
- (i) $(x|x) = 0$ if and only if $x = 0_{\text{the vectors of it}}$,
- (ii) $0 \leq (x|x)$,
- (iii) $(x|y) = (y|x)$,

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(iv) $((x+y)|z) = (x|z) + (y|z)$,
(v) $((a \cdot x)|y) = a \cdot (x|y)$.

We follow the rules: X denotes a real unitary space and x, y, z, u, v denote points of X. We now state a number of propositions:

(1) $(x|x) = 0$ if and only if $x = 0_{\text{the vectors of } X}$.
(2) $0 \leq (x|x)$.
(3) $(x|y) = (y|x)$.
(4) $((x+y)|z) = (x|z) + (y|z)$.
(5) $((a \cdot x)|y) = a \cdot (x|y)$.
(6) $(0_{\text{the vectors of } X}|0_{\text{the vectors of } X}) = 0$.
(7) $(x|(y+z)) = (x|y) + (x|z)$.
(8) $(x|(a \cdot y)) = a \cdot (x|y)$.
(9) $((a \cdot x)|y) = (x|(a \cdot y))$.
(10) $((a \cdot x + b \cdot y)|z) = a \cdot (x|z) + b \cdot (y|z)$.
(11) $(x|(a \cdot y + b \cdot z)) = a \cdot (x|y) + b \cdot (x|z)$.
(12) $((-x)|y) = (x|-y)$.
(13) $((-x)|y) = -(x|y)$.
(14) $(x|-y) = -(x|y)$.
(15) $((-x)|-y) = (x|y)$.
(16) $((x-y)|z) = (x|z) - (y|z)$.
(17) $(x|(y-z)) = (x|y) - (x|z)$.
(18) $((x-y)|(u-v)) = ((x|u) - (x|v) - (y|u)) + (y|v)$.
(19) $(0_{\text{the vectors of } X}|x) = 0$.
(20) $(x|0_{\text{the vectors of } X}) = 0$.
(21) $((x+y)|(x+y)) = (x|x) + 2 \cdot (x|y) + (y|y)$.
(22) $((x+y)|(x-y)) = (x|x) - (y|y)$.
(23) $((x-y)|(x-y)) = ((x|x) - 2 \cdot (x|y)) + (y|y)$.
(24) $|(x|y)| \leq \sqrt{(x|x)} \cdot \sqrt{(y|y)}$.

Let us consider X, x, y. We say that x, y are ortogonal if and only if:

(Def.3) $(x|y) = 0$.

The following propositions are true:

(25) If x, y are ortogonal, then y, x are ortogonal.
(26) If x, y are ortogonal, then x, $-y$ are ortogonal.
(27) If x, y are ortogonal, then $-x$, y are ortogonal.
(28) If x, y are ortogonal, then $-x$, $-y$ are ortogonal.
(29) x, $0_{\text{the vectors of } X}$ are ortogonal.
(30) If x, y are ortogonal, then $((x+y)|(x+y)) = (x|x) + (y|y)$.
(31) If x, y are ortogonal, then $((x-y)|(x-y)) = (x|x) + (y|y)$.

Let us consider X, x. The functor $\|x\|$ yielding a real number is defined by:

(Def.4) $\|x\| = \sqrt{(x|x)}$.

The following propositions are true:

(32) $\|x\| = 0$ if and only if $x = 0_{\text{the vectors of } X}$.

(33) $\|a \cdot x\| = |a| \cdot \|x\|$.

(34) $0 \leq \|x\|$.

(35) $|(x|y)| \leq \|x\| \cdot \|y\|$.

(36) $\|x + y\| \leq \|x\| + \|y\|$.

(37) $\|-x\| = \|x\|$.

(38) $\|x\| - \|y\| \leq \|x - y\|$.

(39) $|\|x\| - \|y\|| \leq \|x - y\|$.

Let us consider X, x, y. The functor $\rho(x, y)$ yielding a real number is defined by:

(Def.5) $\rho(x, y) = \|x - y\|$.

One can prove the following propositions:

(40) $\rho(x, y) = \rho(y, x)$.

(41) $\rho(x, x) = 0$.

(42) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

(43) $x \neq y$ if and only if $\rho(x, y) \neq 0$.

(44) $\rho(x, y) \geq 0$.

(45) $x \neq y$ if and only if $\rho(x, y) > 0$.

(46) $\rho(x, y) = \sqrt{((x - y)|(x - y))}$.

(47) $\rho(x + y, u + v) \leq \rho(x, u) + \rho(y, v)$.

(48) $\rho(x - y, u - v) \leq \rho(x, u) + \rho(y, v)$.

(49) $\rho(x - z, y - z) = \rho(x, y)$.

(50) $\rho(x - z, y - z) \leq \rho(z, x) + \rho(z, y)$.

Let us consider X. A subset of X is a subset of the vectors of the vectors of X.

Let us consider X. A function is called a sequence of X if:

(Def.6) $\operatorname{dom} \mathrm{it} = \mathbb{N}$ and $\operatorname{rng} \mathrm{it} \subseteq$ the vectors of the vectors of X.

For simplicity we adopt the following rules: s_1, s_2, s_3, s_4, s'_1 denote sequences of X, k, n, m denote natural numbers, f denotes a function, and d is arbitrary. We now state four propositions:

(51) f is a sequence of X if and only if $\operatorname{dom} f = \mathbb{N}$ and $\operatorname{rng} f \subseteq$ the vectors of the vectors of X.

(52) f is a sequence of X if and only if $\operatorname{dom} f = \mathbb{N}$ and for every d such that $d \in \mathbb{N}$ holds $f(d)$ is a point of X.

(53) For all s_1, s'_1 such that for every n holds $s_1(n) = s'_1(n)$ holds $s_1 = s'_1$.

(54) For every n holds $s_1(n)$ is a point of X.

Let us consider X, s_1, n. Then $s_1(n)$ is a point of X.

The scheme *Ex_Seq_in_RUS* concerns a real unitary space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$ and states that:

there exists a sequence s_1 of $\mathcal{A}$ such that for every n holds $s_1(n) = \mathcal{F}(n)$
for all values of the parameters.

Let us consider X, s_2, s_3. The functor $s_2 + s_3$ yielding a sequence of X is defined by:

(Def.7) for every n holds $(s_2 + s_3)(n) = s_2(n) + s_3(n)$.

Let us consider X, s_2, s_3. The functor $s_2 - s_3$ yielding a sequence of X is defined as follows:

(Def.8) for every n holds $(s_2 - s_3)(n) = s_2(n) - s_3(n)$.

Let us consider X, s_1, a. The functor $a \cdot s_1$ yields a sequence of X and is defined as follows:

(Def.9) for every n holds $(a \cdot s_1)(n) = a \cdot s_1(n)$.

Let us consider X, s_1. The functor $-s_1$ yields a sequence of X and is defined by:

(Def.10) for every n holds $(-s_1)(n) = -s_1(n)$.

Let us consider X, s_1. We say that s_1 is constant if and only if:

(Def.11) there exists x such that for every n holds $s_1(n) = x$.

Let us consider X, s_1, x. The functor $s_1 + x$ yielding a sequence of X is defined as follows:

(Def.12) for every n holds $(s_1 + x)(n) = s_1(n) + x$.

Let us consider X, s_1, x. The functor $s_1 - x$ yields a sequence of X and is defined by:

(Def.13) for every n holds $(s_1 - x)(n) = s_1(n) - x$.

We now state a number of propositions:

(55) $s_2 + s_3 = s_3 + s_2$.

(56) $s_2 + (s_3 + s_4) = s_2 + s_3 + s_4$.

(57) If s_2 is constant and s_3 is constant and $s_1 = s_2 + s_3$, then s_1 is constant.

(58) If s_2 is constant and s_3 is constant and $s_1 = s_2 - s_3$, then s_1 is constant.

(59) If s_2 is constant and $s_1 = a \cdot s_2$, then s_1 is constant.

(60) For every x there exists s_1 such that $\operatorname{rng} s_1 = \{x\}$.

(61) There exists s_1 such that $\operatorname{rng} s_1 = \{0_{\text{the vectors of } X}\}$.

(62) If there exists x such that for every n holds $s_1(n) = x$, then there exists x such that $\operatorname{rng} s_1 = \{x\}$.

(63) If there exists x such that $\operatorname{rng} s_1 = \{x\}$, then for every n holds $s_1(n) = s_1(n + 1)$.

(64) If for every n holds $s_1(n) = s_1(n + 1)$, then for all n, k holds $s_1(n) = s_1(n + k)$.

(65) If for all n, k holds $s_1(n) = s_1(n+k)$, then for all n, m holds $s_1(n) = s_1(m)$.

(66) If for all n, m holds $s_1(n) = s_1(m)$, then there exists x such that for every n holds $s_1(n) = x$.

(67) s_1 is constant if and only if there exists x such that $\operatorname{rng} s_1 = \{x\}$.

(68) s_1 is constant if and only if for every n holds $s_1(n) = s_1(n+1)$.

(69) s_1 is constant if and only if for all n, k holds $s_1(n) = s_1(n+k)$.

(70) s_1 is constant if and only if for all n, m holds $s_1(n) = s_1(m)$.

(71) $s_2 - s_3 = s_2 + -s_3$.

(72) $s_1 = s_1 + 0_{\text{the vectors of } X}$.

(73) $a \cdot (s_2 + s_3) = a \cdot s_2 + a \cdot s_3$.

(74) $(a+b) \cdot s_1 = a \cdot s_1 + b \cdot s_1$.

(75) $a \cdot b \cdot s_1 = a \cdot (b \cdot s_1)$.

(76) $1 \cdot s_1 = s_1$.

(77) $(-1) \cdot s_1 = -s_1$.

(78) $s_1 - x = s_1 + -x$.

(79) $s_2 - s_3 = -(s_3 - s_2)$.

(80) $s_1 = s_1 - 0_{\text{the vectors of } X}$.

(81) $s_1 = --s_1$.

(82) $s_2 - (s_3 + s_4) = s_2 - s_3 - s_4$.

(83) $(s_2 + s_3) - s_4 = s_2 + (s_3 - s_4)$.

(84) $s_2 - (s_3 - s_4) = (s_2 - s_3) + s_4$.

(85) $a \cdot (s_2 - s_3) = a \cdot s_2 - a \cdot s_3$.

References

[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[6] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[7] Jan Popiołek. Quadratic inequalities. *Formalized Mathematics*, 2(**4**):507–509, 1991.

[8] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.

[9] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[10] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[11] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[13] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[14] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.

[15] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[16] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[17] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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