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Mostowski's Fundamental Operations - Part I

Andrzej Kondracki
Warsaw University

Summary. In the chapter II.4 of his book [17] A.Mostowski introduces what he calls fundamental operations:
$A_1(a,b) = \{\{\langle 0,x\rangle,\langle 1,y\rangle\} : x \in y \wedge x \in a \wedge y \in a\}$,
$A_2(a,b) = \{a,b\}$,
$A_3(a,b) = \bigcup a$,
$A_4(a,b) = \{\{\langle x,y\rangle\} : x \in a \wedge y \in b\}$,
$A_5(a,b) = \{x \cup y : x \in a \wedge y \in b\}$,
$A_6(a,b) = \{x \setminus y : x \in a \wedge y \in b\}$,
$A_7(a,b) = \{x \circ y : x \in a \wedge y \in b\}$.
He proves that if a non-void class is closed under these operations then it is predicatively closed. Then he formulates sufficient criteria for a class to be a model of ZF set theory (theorem 4.12).

The article includes the translation of this part of Mostowski's book. The fundamental operations are defined (to be precise, not these operations, but the notions of closure of a class with respect to them). Some properties of classes closed under these operations are proved. At last it is proved that if a non-void class X is closed under the operations $A_1 - A_7$ then $D_H(a) \in X$ for every a in X and every H being formula of ZF language ($D_H(a)$ consists of all finite sequences with terms belonging to a which satisfy H in a).

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The articles [20], [12], [7], [10], [4], [11], [13], [18], [2], [1], [24], [19], [8], [5], [9], [6], [16], [21], [14], [22], [15], [3], and [23] provide the notation and terminology for this paper. For simplicity we follow the rules: V will be a universal class, a, b, x, y will be elements of V, X will be a subclass of V, o, p, q, r, s, t, u will be arbitrary, A, B will be sets, n will be an element of ω, f_1 will be a finite subset of ω, E will be a non-empty set, f will be a function from VAR into E, k will be a natural number, v_1, v_2 will be elements of VAR, and H, H' will be ZF-formulae. Let us consider A, B. The functor AB yielding a set is defined as follows:

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(Def.1) $p \in AB$ if and only if there exist q, r, s such that $p = \langle q, s\rangle$ and $\langle q, r\rangle \in A$ and $\langle r, s\rangle \in B$.

Let us consider V, x, y. Then xy is an element of V.

The function decode from ω into VAR is defined by:

(Def.2) for every p such that $p \in \omega$ holds $\mathrm{decode}(p) = x_{\mathrm{card}\, p}$.

Let us consider v_1. The functor ${}^{v_1}x$ yielding a natural number is defined by:

(Def.3) $x_{{}^{v_1}x} = v_1$.

Let A be a finite subset of VAR. The functor $\mathrm{code}(A)$ yielding a finite subset of ω is defined as follows:

(Def.4) $\mathrm{code}(A) = (\mathrm{decode}^{-1}) \circ A$.

Let us consider H. Then $\mathrm{Free}\, H$ is a finite subset of VAR.

Let us consider v_1. Then $\{v_1\}$ is a finite subset of VAR. Let us consider v_2. Then $\{v_1, v_2\}$ is a finite subset of VAR.

Let us consider H, E. The functor $\mathrm{D}_E(H)$ yielding a set is defined by:

(Def.5) $p \in \mathrm{D}_E(H)$ if and only if there exists f such that $p = (f \cdot \mathrm{decode}) \restriction \mathrm{code}(\mathrm{Free}\, H)$ and $f \in \mathrm{St}_E(H)$.

Let us consider n. Then $\{n\}$ is a finite subset of ω.

We now define several new predicates. Let us consider V, X. We say that X is closed w.r.t. A1 if and only if:

(Def.6) for every a such that $a \in X$ holds $\{\{\langle \mathbf{0}_V, x\rangle, \langle \mathbf{1}_V, y\rangle\} : x \in y \wedge x \in a \wedge y \in a\} \in X$.

We say that X is closed w.r.t. A2 if and only if:

(Def.7) for all a, b such that $a \in X$ and $b \in X$ holds $\{a, b\} \in X$.

We say that X is closed w.r.t. A3 if and only if:

(Def.8) for every a such that $a \in X$ holds $\bigcup a \in X$.

We say that X is closed w.r.t. A4 if and only if:

(Def.9) for all a, b such that $a \in X$ and $b \in X$ holds $\{\{\langle x, y\rangle\} : x \in a \wedge y \in b\} \in X$.

We say that X is closed w.r.t. A5 if and only if:

(Def.10) for all a, b such that $a \in X$ and $b \in X$ holds $\{x \cup y : x \in a \wedge y \in b\} \in X$.

We say that X is closed w.r.t. A6 if and only if:

(Def.11) for all a, b such that $a \in X$ and $b \in X$ holds $\{x \setminus y : x \in a \wedge y \in b\} \in X$.

We say that X is closed w.r.t. A7 if and only if:

(Def.12) for all a, b such that $a \in X$ and $b \in X$ holds $\{xy : x \in a \wedge y \in b\} \in X$.

Let us consider V, X. We say that X is closed w.r.t. A1-A7 if and only if:

(Def.13) X is closed w.r.t. A1 and X is closed w.r.t. A2 and X is closed w.r.t. A3 and X is closed w.r.t. A4 and X is closed w.r.t. A5 and X is closed w.r.t. A6 and X is closed w.r.t. A7.

We now state a number of propositions:

(1) $X \subseteq V$ but if $o \in X$, then o is an element of V but if $o \in A$ and $A \in X$, then o is an element of V.

(2) If X is closed w.r.t. A1-A7, then $o \in X$ if and only if $\{o\} \in X$ but if $A \in X$, then $\bigcup A \in X$.

(3) If X is closed w.r.t. A1-A7, then $\emptyset \in X$ and $\mathbf{0} \in X$.

(4) If X is closed w.r.t. A1-A7 and $A \in X$ and $B \in X$, then $A \cup B \in X$ and $A \setminus B \in X$ and $AB \in X$.

(5) If X is closed w.r.t. A1-A7 and $A \in X$ and $B \in X$, then $A \cap B \in X$.

(6) If X is closed w.r.t. A1-A7 and $o \in X$ and $p \in X$, then $\{o, p\} \in X$ and $\langle o, p\rangle \in X$.

(7) If X is closed w.r.t. A1-A7, then $\omega \subseteq X$.

(8) If X is closed w.r.t. A1-A7, then $\omega^{f_1} \subseteq X$.

(9) If X is closed w.r.t. A1-A7 and $a \in X$, then $a^{f_1} \in X$.

(10) If X is closed w.r.t. A1-A7 and $a \in \omega^{f_1}$ and $b \in X$, then $\{ax : x \in b\} \in X$.

(11) If X is closed w.r.t. A1-A7 and $n \in f_1$ and $a \in X$ and $b \in X$ and $b \subseteq a^{f_1}$, then $\{x : x \in a^{f_1 \setminus \{n\}} \wedge \bigvee_u \{\langle n, u\rangle\} \cup x \in b\} \in X$.

(12) If X is closed w.r.t. A1-A7 and $n \notin f_1$ and $a \in X$ and $b \in X$ and $b \subseteq a^{f_1}$, then $\{\{\langle n, x\rangle\} \cup y : x \in a \wedge y \in b\} \in X$.

(13) If X is closed w.r.t. A1-A7 and B is finite and for every o such that $o \in B$ holds $o \in X$, then $B \in X$.

(14) If X is closed w.r.t. A1-A7 and $A \subseteq X$ and $y \in A^{f_1}$, then $y \in X$.

(15) If X is closed w.r.t. A1-A7 and $n \notin f_1$ and $a \in X$ and $a \subseteq X$ and $y \in a^{f_1}$, then $\{\{\langle n, x\rangle\} \cup y : x \in a\} \in X$.

(16) Suppose X is closed w.r.t. A1-A7 and $n \notin f_1$ and $a \in X$ and $a \subseteq X$ and $y \in a^{f_1}$ and $b \subseteq a^{f_1 \cup \{n\}}$ and $b \in X$. Then $\{x : x \in a \wedge \{\langle n, x\rangle\} \cup y \in b\} \in X$.

(17) If X is closed w.r.t. A1-A7 and $a \in X$, then $\{\{\langle \mathbf{0}_V, x\rangle, \langle \mathbf{1}_V, x\rangle\} : x \in a\} \in X$.

(18) If X is closed w.r.t. A1-A7 and $E \in X$, then for all v_1, v_2 holds $\mathrm{D}_E(v_1{=}v_2) \in X$ and $\mathrm{D}_E(v_1 \epsilon v_2) \in X$.

(19) If X is closed w.r.t. A1-A7 and $E \in X$, then for every H such that $\mathrm{D}_E(H) \in X$ holds $\mathrm{D}_E(\neg H) \in X$.

(20) If X is closed w.r.t. A1-A7 and $E \in X$, then for all H, H' such that $\mathrm{D}_E(H) \in X$ and $\mathrm{D}_E(H') \in X$ holds $\mathrm{D}_E(H \wedge H') \in X$.

(21) If X is closed w.r.t. A1-A7 and $E \in X$, then for all H, v_1 such that $\mathrm{D}_E(H) \in X$ holds $\mathrm{D}_E(\forall_{v_1} H) \in X$.

(22) If X is closed w.r.t. A1-A7 and $E \in X$, then $\mathrm{D}_E(H) \in X$.

(23) If X is closed w.r.t. A1-A7, then $n \in X$ and $\mathbf{0}_V \in X$ and $\mathbf{1}_V \in X$.

(24) $\{\langle o, p\rangle, \langle p, p\rangle\}\{\langle p, q\rangle\} = \{\langle o, q\rangle, \langle p, q\rangle\}$.

(25) If $p \neq r$, then $\{\langle o, p\rangle, \langle q, r\rangle\}\{\langle p, s\rangle, \langle r, t\rangle\} = \{\langle o, s\rangle, \langle q, t\rangle\}$.

(26) ${}^{x_k}x = k$.

(27) $\mathrm{code}(\{v_1\}) = \{\mathrm{ord}({}^{v_1}x)\}$ and $\mathrm{code}(\{v_1, v_2\}) = \{\mathrm{ord}({}^{v_1}x), \mathrm{ord}({}^{v_2}x)\}$.

(28) $\mathrm{dom}\, f = \{o, q\}$ if and only if $\mathrm{graph}\, f = \{\langle o, f(o)\rangle, \langle q, f(q)\rangle\}$.

(29) $\mathrm{dom}\,\mathrm{decode} = \omega$ and $\mathrm{rng}\,\mathrm{decode} = \mathrm{VAR}$ and decode is one-to-one and decode^{-1} is one-to-one and $\mathrm{dom}(\mathrm{decode}^{-1}) = \mathrm{VAR}$ and $\mathrm{rng}(\mathrm{decode}^{-1}) = \omega$.

(30) For every finite subset A of VAR holds $A \approx \mathrm{code}(A)$.

(31) If $A \in \omega$, then $A = \mathrm{ord}(\mathrm{card}\, A)$ and $A = \mathrm{ord}({}^{x_{\mathrm{card}\, A}}x)$.

One can prove the following propositions:

(32) $\mathrm{dom}((f \cdot \mathrm{decode}) \restriction f_1) = f_1$ and $\mathrm{rng}((f \cdot \mathrm{decode}) \restriction f_1) \subseteq E$ and $(f \cdot \mathrm{decode}) \restriction f_1 \in E^{f_1}$ and $\mathrm{dom}(f \cdot \mathrm{decode}) = \omega$ and $\mathrm{rng}(f \cdot \mathrm{decode}) \subseteq E$.

(33) $\mathrm{decode}(\mathrm{ord}({}^{v_1}x)) = v_1$ and $\mathrm{decode}^{-1}(v_1) = \mathrm{ord}({}^{v_1}x)$ and $(f \cdot \mathrm{decode})(\mathrm{ord}({}^{v_1}x)) = f(v_1)$.

(34) For every finite subset A of VAR holds $p \in \mathrm{code}(A)$ if and only if there exists v_1 such that $v_1 \in A$ and $p = \mathrm{ord}({}^{v_1}x)$.

(35) For all finite subsets A, B of VAR holds $\mathrm{code}(A \cup B) = \mathrm{code}(A) \cup \mathrm{code}(B)$ and $\mathrm{code}(A \setminus B) = \mathrm{code}(A) \setminus \mathrm{code}(B)$.

(36) If $v_1 \in \mathrm{Free}\, H$, then $((f \cdot \mathrm{decode}) \restriction \mathrm{code}(\mathrm{Free}\, H))(\mathrm{ord}({}^{v_1}x)) = f(v_1)$.

(37) For all functions f, g from VAR into E such that $(f \cdot \mathrm{decode}) \restriction \mathrm{code}(\mathrm{Free}\, H) = (g \cdot \mathrm{decode}) \restriction \mathrm{code}(\mathrm{Free}\, H)$ and $f \in \mathrm{St}_E(H)$ holds $g \in \mathrm{St}_E(H)$.

(38) If $p \in E^{f_1}$, then there exists f such that $p = (f \cdot \mathrm{decode}) \restriction f_1$.

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