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The σ-additive Measure Theory

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Summary. The article contains a definition and basic properties of a σ-additive, nonnegative measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ - by [11]. We present definitions of σ-field of sets, σ-additive measure, measurable sets, measure zero sets and the basic theorems describing relationships between the notions mentioned above. The work is the third part of the series of articles concerning the Lebesgue measure theory.

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The papers [13], [12], [7], [8], [5], [6], [1], [10], [2], [9], [3], and [4] provide the terminology and notation for this paper. One can prove the following four propositions:

(1) For all sets X, Y holds $\bigcup\{X, Y, \emptyset\} = \bigcup\{X, Y\}$.

(2) For every natural number n holds $n = 0$ or $n = 1$ or $1 < n$.

(4)[1] For all *Real numbers* x, y, s, t such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq s$ and $x \leq y$ and $s \leq t$ holds $x + s \leq y + t$.

(5) For all *Real numbers* x, y, z such that $0_{\overline{\mathbb{R}}} \leq y$ and $0_{\overline{\mathbb{R}}} \leq z$ and $x = y + z$ and $y < +\infty$ holds $z = x - y$.

Let X be a set. A set is called a non-empty family of subsets of X if:

(Def.1) it $\neq \emptyset$ and for an arbitrary A such that $A \in$ it holds $A \in 2^X$.

One can prove the following propositions:

(6) For every set X and for every subset A of X holds $\{A\}$ is a non-empty family of subsets of X.

(7) For every set X and for all subsets A, B of X holds $\{A, B\}$ is a non-empty family of subsets of X.

(8) For every set X and for all subsets A, B, C of X holds $\{A, B, C\}$ is a non-empty family of subsets of X.

[1] The proposition (3) was either repeated or obvious.

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(9) For every set X holds $\{\emptyset\}$ is a non-empty family of subsets of X.

(10) For every set X holds $\{\emptyset, X\}$ is a non-empty family of subsets of X.

(12)[2] For every set X holds 2^X is a non-empty family of subsets of X.

The scheme *DomsetFamEx* concerns a set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a non-empty family F of subsets of $\mathcal{A}$ such that for every set B holds $B \in F$ if and only if $B \subseteq \mathcal{A}$ and $\mathcal{P}[B]$

provided the following condition is satisfied:

- there exists a set B such that $B \subseteq \mathcal{A}$ and $\mathcal{P}[B]$.

Let X be a set, and let S be a non-empty family of subsets of X. The functor $X \setminus S$ yielding a non-empty family of subsets of X is defined as follows:

(Def.2) for every set A holds $A \in X \setminus S$ if and only if there exists a set B such that $B \in S$ and $A = X \setminus B$.

We now state three propositions:

(13) For every set X and for every non-empty family S of subsets of X and for every set A holds $A \in X \setminus S$ if and only if there exists a set B such that $B \in S$ and $A = X \setminus B$.

(14) For every set X and for every non-empty family S of subsets of X holds $S = X \setminus (X \setminus S)$.

(15) For every set X and for every non-empty family S of subsets of X holds $\bigcap S = X \setminus \bigcup(X \setminus S)$ and $\bigcup S = X \setminus \bigcap(X \setminus S)$.

Let X be a set. A non-empty family of subsets of X is said to be a field of subsets of X if:

(Def.3) for every set A such that $A \in$ it holds $X \setminus A \in$ it and for all sets A, B such that $A \in$ it and $B \in$ it holds $A \cup B \in$ it.

The following propositions are true:

(17)[3] For every set X and for every field S of subsets of X holds $S = X \setminus S$.

(18) For every set X and for an arbitrary M holds M is a field of subsets of X if and only if there exists a non-empty family S of subsets of X such that $M = S$ and for every set A such that $A \in S$ holds $X \setminus A \in S$ and for all sets A, B such that $A \in S$ and $B \in S$ holds $A \cup B \in S$.

(19) For every set X and for every non-empty family S of subsets of X holds S is a field of subsets of X if and only if for every set A such that $A \in S$ holds $X \setminus A \in S$ and for all sets A, B such that $A \in S$ and $B \in S$ holds $A \cap B \in S$.

(20) For every set X and for every field S of subsets of X and for all sets A, B such that $A \in S$ and $B \in S$ holds $A \setminus B \in S$.

(21) For every set X and for every field S of subsets of X holds $\emptyset \in S$ and $X \in S$.

[2]The proposition (11) was either repeated or obvious.

[3]The proposition (16) was either repeated or obvious.

Let X be a set, and let S be a non-empty family of subsets of X, and let F be a function from S into $\overline{\mathbb{R}}$, and let A be an element of S. Then $F(A)$ is a *Real number*.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$, and let n be a natural number. Then $F(n)$ is a *Real number*.

Let X be a set, and let S be a non-empty family of subsets of X, and let F be a function from S into $\overline{\mathbb{R}}$. We say that F is non-negative if and only if:

(Def.4) for every element A of S holds $0_{\overline{\mathbb{R}}} \leq F(A)$.

We now state the proposition

(23)[4] For every set X and for every field S of subsets of X there exists a function M from S into $\overline{\mathbb{R}}$ such that M is non-negative and $M(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for all elements A, B of S such that $A \cap B = \emptyset$ holds $M(A \cup B) = M(A) + M(B)$.

Let X be a set, and let S be a field of subsets of X. A function from S into $\overline{\mathbb{R}}$ is called a measure on S if:

(Def.5) it is non-negative and $\text{it}(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for all elements A, B of S such that $A \cap B = \emptyset$ holds $\text{it}(A \cup B) = \text{it}(A) + \text{it}(B)$.

Next we state two propositions:

(25)[5] For every set X and for every field S of subsets of X and for every measure M on S and for all elements A, B of S such that $A \subseteq B$ holds $M(A) \leq M(B)$.

(26) For every set X and for every field S of subsets of X and for every measure M on S and for all elements A, B of S such that $A \subseteq B$ and $M(A) < +\infty$ holds $M(B \setminus A) = M(B) - M(A)$.

Let X be a set, and let S be a field of subsets of X, and let A, B be elements of S. Then $A \cup B$ is an element of S.

Let X be a set, and let S be a field of subsets of X, and let A, B be elements of S. Then $A \cap B$ is an element of S.

Let X be a set, and let S be a field of subsets of X, and let A, B be elements of S. Then $A \setminus B$ is an element of S.

The following proposition is true

(27) For every set X and for every field S of subsets of X and for every measure M on S and for all elements A, B of S holds $M(A \cup B) \leq M(A) + M(B)$.

Let X be a set, and let S be a field of subsets of X, and let M be a measure on S, and let A be a set. We say that A is measurable w.r.t. M if and only if:

(Def.6) $A \in S$.

The following proposition is true

[4]The proposition (22) was either repeated or obvious.
[5]The proposition (24) was either repeated or obvious.

(29)[6] For every set X and for every field S of subsets of X and for every measure M on S holds $\emptyset$ is measurable w.r.t. M and X is measurable w.r.t. M and for all sets A, B such that A is measurable w.r.t. M and B is measurable w.r.t. M holds $X \setminus A$ is measurable w.r.t. M and $A \cup B$ is measurable w.r.t. M and $A \cap B$ is measurable w.r.t. M.

Let X be a set, and let S be a field of subsets of X, and let M be a measure on S. An element of S is called a set of measure zero w.r.t. M if:

(Def.7) $M(\mathrm{it}) = 0_{\overline{\mathbb{R}}}$.

The following propositions are true:

(31)[7] For every set X and for every field S of subsets of X and for every measure M on S and for every element A of S and for every set B of measure zero w.r.t. M such that $A \subseteq B$ holds A is a set of measure zero w.r.t. M.

(32) For every set X and for every field S of subsets of X and for every measure M on S and for all sets A, B of measure zero w.r.t. M holds $A \cup B$ is a set of measure zero w.r.t. M and $A \cap B$ is a set of measure zero w.r.t. M and $A \setminus B$ is a set of measure zero w.r.t. M.

(33) For every set X and for every field S of subsets of X and for every measure M on S and for every element A of S and for every set B of measure zero w.r.t. M holds $M(A \cup B) = M(A)$ and $M(A \cap B) = 0_{\overline{\mathbb{R}}}$ and $M(A \setminus B) = M(A)$.

(34) For every set X and for every subset A of X there exists a function F from $\mathbb{N}$ into 2^X such that $\mathrm{rng}\, F = \{A\}$.

(35) For every set X and for every subset A of X there exists a function F from $\mathbb{N}$ into $\{A\}$ such that for every natural number n holds $F(n) = A$.

Let X be a set. A non-empty family of subsets of X is said to be a denumerable family of subsets of X if:

(Def.8) there exists a function F from $\mathbb{N}$ into 2^X such that $\mathrm{it} = \mathrm{rng}\, F$.

We now state several propositions:

(37)[8] For every set X and for every denumerable family S of subsets of X there exists a function F from $\mathbb{N}$ into 2^X such that $S = \mathrm{rng}\, F$.

(38) For every set X and for every subsets A, B, C of X there exists a function F from $\mathbb{N}$ into 2^X such that $\mathrm{rng}\, F = \{A, B, C\}$ and $F(0) = A$ and $F(1) = B$ and for every natural number n such that $1 < n$ holds $F(n) = C$.

(39) For every set X and for all subsets A, B of X holds $\{A, B, \emptyset\}$ is a denumerable family of subsets of X.

[6]The proposition (28) was either repeated or obvious.
[7]The proposition (30) was either repeated or obvious.
[8]The proposition (36) was either repeated or obvious.

(40) For every set X and for every subsets A, B of X there exists a function F from $\mathbb{N}$ into 2^X such that $\operatorname{rng} F = \{A, B\}$ and $F(0) = A$ and for every natural number n such that $0 < n$ holds $F(n) = B$.

(41) For every set X and for all subsets A, B of X holds $\{A, B\}$ is a denumerable family of subsets of X.

(42) For every set X and for every denumerable family S of subsets of X holds $X \setminus S$ is a denumerable family of subsets of X.

Let X be a set. A non-empty family of subsets of X is said to be a σ-field of subsets of X if:

(Def.9) for every set A such that $A \in$ it holds $X \setminus A \in$ it and for every denumerable family M of subsets of X such that $M \subseteq$ it holds $\bigcup M \in$ it.

One can prove the following propositions:

(44)[9] For every set X and for every non-empty family S of subsets of X such that S is a σ-field of subsets of X holds S is a field of subsets of X.

(45) For every set X and for every σ-field S of subsets of X holds $\emptyset \in S$ and $X \in S$.

(46) For every set X and for every σ-field S of subsets of X and for all sets A, B such that $A \in S$ and $B \in S$ holds $A \cup B \in S$ and $A \cap B \in S$.

(47) For every set X and for every σ-field S of subsets of X and for all sets A, B such that $A \in S$ and $B \in S$ holds $A \setminus B \in S$.

(48) For every set X and for every σ-field S of subsets of X holds $S = X \setminus S$.

(49) For every set X and for every non-empty family S of subsets of X holds S is a σ-field of subsets of X if and only if for every set A such that $A \in S$ holds $X \setminus A \in S$ and for every denumerable family M of subsets of X such that $M \subseteq S$ holds $\bigcap M \in S$.

Let X be a set, and let S be a σ-field of subsets of X. A function from $\mathbb{N}$ into S is said to be a sequence of separated subsets of S if:

(Def.10) for all natural numbers n, m such that $n \neq m$ holds $\mathrm{it}(n) \cap \mathrm{it}(m) = \emptyset$.

We now state the proposition

(51)[10] For every set X and for every σ-field S of subsets of X and for every function F from $\mathbb{N}$ into S and for every function M from S into $\overline{\mathbb{R}}$ holds $M \cdot F$ is a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$.

Let X be a set, and let S be a σ-field of subsets of X, and let F be a function from $\mathbb{N}$ into S. Then $\operatorname{rng} F$ is a non-empty family of subsets of X.

Let X be a set, and let S be a σ-field of subsets of X, and let F be a function from $\mathbb{N}$ into S, and let M be a function from S into $\overline{\mathbb{R}}$. Then $M \cdot F$ is a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$.

Next we state several propositions:

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[9]The proposition (43) was either repeated or obvious.

[10]The proposition (50) was either repeated or obvious.

(52) For every set X and for every σ-field S of subsets of X and for every function F from $\mathbb{N}$ into S holds $\operatorname{rng} F$ is a denumerable family of subsets of X.

(53) For every set X and for every σ-field S of subsets of X and for every function F from $\mathbb{N}$ into S holds $\bigcup \operatorname{rng} F$ is an element of S.

(54) For every set X and for every σ-field S of subsets of X and for every function F from $\mathbb{N}$ into S and for every function M from S into $\overline{\mathbb{R}}$ such that M is non-negative holds $M \cdot F$ is non-negative.

(55) For every set X and for every σ-field S of subsets of X and for every *Real numbers* a, b there exists a function M from S into $\overline{\mathbb{R}}$ such that for every element A of S holds if $A = \emptyset$, then $M(A) = a$ but if $A \neq \emptyset$, then $M(A) = b$.

(56) For every set X and for every σ-field S of subsets of X there exists a function M from S into $\overline{\mathbb{R}}$ such that for every element A of S holds if $A = \emptyset$, then $M(A) = 0_{\overline{\mathbb{R}}}$ but if $A \neq \emptyset$, then $M(A) = +\infty$.

(57) For every set X and for every σ-field S of subsets of X there exists a function M from S into $\overline{\mathbb{R}}$ such that for every element A of S holds $M(A) = 0_{\overline{\mathbb{R}}}$.

(58) For every set X and for every σ-field S of subsets of X there exists a function M from S into $\overline{\mathbb{R}}$ such that M is non-negative and $M(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for every sequence F of separated subsets of S holds $\sum(M \cdot F) = M(\bigcup \operatorname{rng} F)$.

Let X be a set, and let S be a σ-field of subsets of X. A function from S into $\overline{\mathbb{R}}$ is said to be a σ-measure on S if:

(Def.11) it is non-negative and $\mathrm{it}(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for every sequence F of separated subsets of S holds $\sum(\mathrm{it} \cdot F) = \mathrm{it}(\bigcup \operatorname{rng} F)$.

Let X be a set. We see that the σ-field of subsets of X is a field of subsets of X.

One can prove the following propositions:

(60)[11] For every set X and for every σ-field S of subsets of X and for every σ-measure M on S holds M is a measure on S.

(61) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for all elements A, B of S such that $A \cap B = \emptyset$ holds $M(A \cup B) = M(A) + M(B)$.

(62) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for all elements A, B of S such that $A \subseteq B$ holds $M(A) \leq M(B)$.

(63) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for all elements A, B of S such that $A \subseteq B$ and $M(A) < +\infty$ holds $M(B \setminus A) = M(B) - M(A)$.

[11]The proposition (59) was either repeated or obvious.

(64) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for all elements A, B of S holds $M(A \cup B) \leq M(A) + M(B)$.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S, and let A be a set. We say that A is measurable w.r.t. M if and only if:

(Def.12) $A \in S$.

Next we state two propositions:

(66)[12] For every set X and for every σ-field S of subsets of X and for every σ-measure M on S holds $\emptyset$ is measurable w.r.t. M and X is measurable w.r.t. M and for all sets A, B such that A is measurable w.r.t. M and B is measurable w.r.t. M holds $X \setminus A$ is measurable w.r.t. M and $A \cup B$ is measurable w.r.t. M and $A \cap B$ is measurable w.r.t. M.

(67) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every denumerable family T of subsets of X such that for every set A such that $A \in T$ holds A is measurable w.r.t. M holds $\bigcup T$ is measurable w.r.t. M and $\bigcap T$ is measurable w.r.t. M.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S. An element of S is called a set of measure zero w.r.t. M if:

(Def.13) $M(\mathrm{it}) = 0_{\overline{\mathbb{R}}}$.

Next we state three propositions:

(69)[13] For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every element A of S and for every set B of measure zero w.r.t. M such that $A \subseteq B$ holds A is a set of measure zero w.r.t. M.

(70) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for all sets A, B of measure zero w.r.t. M holds $A \cup B$ is a set of measure zero w.r.t. M and $A \cap B$ is a set of measure zero w.r.t. M and $A \setminus B$ is a set of measure zero w.r.t. M.

(71) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every element A of S and for every set B of measure zero w.r.t. M holds $M(A \cup B) = M(A)$ and $M(A \cap B) = 0_{\overline{\mathbb{R}}}$ and $M(A \setminus B) = M(A)$.

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