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Real Function Uniform Continuity [1]

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Summary. The uniform continuity for real functions is introduced. More theorems concerning continuous functions are given. (See [10]) The Darboux Theorem is exposed. Algebraic features for uniformly continuous functions are presented. Various facts, e.g., a continuous function on a compact set is uniformly continuous are proved.

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The notation and terminology used in this paper have been introduced in the following articles: [12], [13], [3], [1], [9], [8], [4], [2], [5], [6], [7], [11], and [10]. For simplicity we adopt the following convention: X, X_1, Z, Z_1 are sets, s, g, r, p, x_1, x_2 are real numbers, Y is a subset of $\mathbb{R}$, and f, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$. Let us consider f, X. We say that f is uniformly continuous on X if and only if:

$X \subseteq \operatorname{dom} f$ and for every r such that $0 < r$ there exists s such that $0 < s$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ and $|x_1 - x_2| < s$ holds $|f(x_1) - f(x_2)| < r$.

We now state a number of propositions:

(1) Given f, X. Then f is uniformly continuous on X if and only if $X \subseteq \operatorname{dom} f$ and for every r such that $0 < r$ there exists s such that $0 < s$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ and $|x_1 - x_2| < s$ holds $|f(x_1) - f(x_2)| < r$.

(2) If f is uniformly continuous on X and $X_1 \subseteq X$, then f is uniformly continuous on X_1.

(3) If f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1, then $f_1 + f_2$ is uniformly continuous on $X \cap X_1$.

(4) If f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1, then $f_1 - f_2$ is uniformly continuous on $X \cap X_1$.

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(5) If f is uniformly continuous on X, then $p \diamond f$ is uniformly continuous on X.

(6) If f is uniformly continuous on X, then $-f$ is uniformly continuous on X.

(7) If f is uniformly continuous on X, then $|f|$ is uniformly continuous on X.

(8) If f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1 and f_1 is bounded on Z and f_2 is bounded on Z_1, then $f_1 \diamond f_2$ is uniformly continuous on $((X \cap Z) \cap X_1) \cap Z_1$.

(9) If f is uniformly continuous on X, then f is continuous on X.

(10) If f is Lipschitzian on X, then f is uniformly continuous on X.

(11) For all f, Y such that Y is compact and f is continuous on Y holds f is uniformly continuous on Y.

(12) For every f such that $\mathrm{dom}\, f$ is compact and f is continuous on $\mathrm{dom}\, f$ holds f is uniformly continuous on $\mathrm{dom}\, f$.

(13) If $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is uniformly continuous on Y, then $f^\circ Y$ is compact.

(14) For all f, Y such that $Y \neq \emptyset$ and $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is uniformly continuous on Y there exist x_1, x_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $f(x_1) = \sup(f^\circ Y)$ and $f(x_2) = \inf(f^\circ Y)$.

(15) If $X \subseteq \mathrm{dom}\, f$ and f is a constant on X, then f is uniformly continuous on X.

(16) If $p \leq g$ and f is continuous on $[p, g]$, then for every r such that $r \in [f(p), f(g)] \cup [f(g), f(p)]$ there exists s such that $s \in [p, g]$ and $r = f(s)$.

(17) If $p \leq g$ and f is continuous on $[p, g]$, then for every r such that $r \in [\inf(f^\circ [p, g]), \sup(f^\circ [p, g])]$ there exists s such that $s \in [p, g]$ and $r = f(s)$.

(18) If f is one-to-one and $p \leq g$ and f is continuous on $[p, g]$, then f is increasing on $[p, g]$ or f is decreasing on $[p, g]$.

(19) Suppose f is one-to-one and $p \leq g$ and f is continuous on $[p, g]$. Then $\inf(f^\circ [p, g]) = f(p)$ and $\sup(f^\circ [p, g]) = f(g)$ or $\inf(f^\circ [p, g]) = f(g)$ and $\sup(f^\circ [p, g]) = f(p)$.

(20) If $p \leq g$ and f is continuous on $[p, g]$, then $f^\circ [p, g] = [\inf(f^\circ [p, g]), \sup(f^\circ [p, g])]$.

(21) If f is one-to-one and $p \leq g$ and f is continuous on $[p, g]$, then f^{-1} is continuous on $[\inf(f^\circ [p, g]), \sup(f^\circ [p, g])]$.

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