FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Monotone Real Sequences. Subsequences

Jarosław Kotowicz[1]
Warsaw University, Białystok

Summary. The article contains definitions of constant, increasing, decreasing, non decreasing, non increasing sequences, the definition of a subsequence and their basic properties.

MML Identifier: SEQM_3.

The articles [2], [4], [3], [1], and [5] provide the terminology and notation for this paper. We adopt the following convention: n, m, k will be natural numbers, r will be a real number, and seq, seq_1, seq_2 will be sequences of real numbers. We now define five new predicates. Let us consider seq. We say that seq is increasing if and only if:

for every n holds $seq(n) < seq(n+1)$.

We say that seq is decreasing if and only if:

for every n holds $seq(n+1) < seq(n)$.

We say that seq is non-decreasing if and only if:

for every n holds $seq(n) \leq seq(n+1)$.

We say that seq is non-increasing if and only if:

for every n holds $seq(n+1) \leq seq(n)$.

We say that seq is constant if and only if:

there exists r such that for every n holds $seq(n) = r$.

Let us consider seq. We say that seq is monotone if and only if:

seq is non-decreasing or seq is non-increasing.

We now state a number of propositions:

(1) seq is increasing if and only if for every n holds $seq(n) < seq(n+1)$.

(2) seq is decreasing if and only if for every n holds $seq(n+1) < seq(n)$.

(3) seq is non-decreasing if and only if for every n holds $seq(n) \leq seq(n+1)$.

(4) seq is non-increasing if and only if for every n holds $seq(n+1) \leq seq(n)$.

[1]Supported by RPBP III.24

ISSN 0777–4028

(5) *seq* is constant if and only if there exists r such that for every n holds $seq(n) = r$.

(6) *seq* is monotone if and only if *seq* is non-decreasing or *seq* is non-increasing.

(7) *seq* is increasing if and only if for all n, m such that $n < m$ holds $seq(n) < seq(m)$.

(8) *seq* is increasing if and only if for all n, k holds $seq(n) < seq((n+1)+k)$.

(9) *seq* is decreasing if and only if for all n, k holds $seq((n+1)+k) < seq(n)$.

(10) *seq* is decreasing if and only if for all n, m such that $n < m$ holds $seq(m) < seq(n)$.

(11) *seq* is non-decreasing if and only if for all n, k holds $seq(n) \leq seq(n+k)$.

(12) *seq* is non-decreasing if and only if for all n, m such that $n \leq m$ holds $seq(n) \leq seq(m)$.

(13) *seq* is non-increasing if and only if for all n, k holds $seq(n+k) \leq seq(n)$.

(14) *seq* is non-increasing if and only if for all n, m such that $n \leq m$ holds $seq(m) \leq seq(n)$.

(15) *seq* is constant if and only if there exists r such that $\operatorname{rng} seq = \{r\}$.

(16) *seq* is constant if and only if for every n holds $seq(n) = seq(n+1)$.

(17) *seq* is constant if and only if for all n, k holds $seq(n) = seq(n+k)$.

(18) *seq* is constant if and only if for all n, m holds $seq(n) = seq(m)$.

(19) If *seq* is increasing, then for every n such that $0 < n$ holds $seq(0) < seq(n)$.

(20) If *seq* is decreasing, then for every n such that $0 < n$ holds $seq(n) < seq(0)$.

(21) If *seq* is non-decreasing, then for every n holds $seq(0) \leq seq(n)$.

(22) If *seq* is non-increasing, then for every n holds $seq(n) \leq seq(0)$.

(23) If *seq* is increasing, then *seq* is non-decreasing.

(24) If *seq* is decreasing, then *seq* is non-increasing.

(25) If *seq* is constant, then *seq* is non-decreasing.

(26) If *seq* is constant, then *seq* is non-increasing.

(27) If *seq* is non-decreasing and *seq* is non-increasing, then *seq* is constant.

A sequence of real numbers is said to be an increasing sequence of naturals if:

$\operatorname{rng} \mathrm{it} \subseteq \mathbb{N}$ and for every n holds $\mathrm{it}(n) < \mathrm{it}(n+1)$.

Let us consider *seq*, k. The functor $seq \frown k$ yielding a sequence of real numbers, is defined as follows:

for every n holds $(seq \frown k)(n) = seq(n+k)$.

In the sequel $Nseq$, $Nseq_1$ will be increasing sequences of naturals. Next we state four propositions:

(28) *seq* is an increasing sequence of naturals if and only if $\operatorname{rng} seq \subseteq \mathbb{N}$ and for every n holds $seq(n) < seq(n+1)$.

(29) *seq* is an increasing sequence of naturals if and only if *seq* is increasing and for every n holds $seq(n)$ is a natural number.

(30) $seq_1 = seq \frown k$ if and only if for every n holds $seq_1(n) = seq(n + k)$.

(31) For every n holds $(seq \cdot Nseq)(n) = seq(Nseq(n))$.

Let us consider $Nseq$, n. Then $Nseq(n)$ is a natural number.

Let us consider $Nseq$, seq. Then $seq \cdot Nseq$ is a sequence of real numbers.

Let us consider $Nseq$, $Nseq_1$. Then $Nseq_1 \cdot Nseq$ is an increasing sequence of naturals.

Let us consider $Nseq$, k. Then $Nseq \frown k$ is an increasing sequence of naturals.

Let us consider seq, seq_1. We say that seq is a subsequence of seq_1 if and only if:

there exists $Nseq$ such that $seq = seq_1 \cdot Nseq$.

Next we state a number of propositions:

(32) seq is a subsequence of seq_1 if and only if there exists $Nseq$ such that $seq = seq_1 \cdot Nseq$.

(33) For every n holds $n \leq Nseq(n)$.

(34) $seq \frown 0 = seq$.

(35) $(seq \frown k) \frown m = (seq \frown m) \frown k$.

(36) $(seq \frown k) \frown m = seq \frown (k + m)$.

(37) $(seq + seq_1) \frown k = seq \frown k + seq_1 \frown k$.

(38) $(-seq) \frown k = -seq \frown k$.

(39) $(seq - seq_1) \frown k = seq \frown k - seq_1 \frown k$.

(40) If seq is non-zero, then $seq \frown k$ is non-zero.

(41) If seq is non-zero, then $seq^{-1} \frown k = (seq \frown k)^{-1}$.

(42) $(seq \cdot seq_1) \frown k = (seq \frown k) \cdot (seq_1 \frown k)$.

(43) If seq_1 is non-zero, then $\frac{seq}{seq_1} \frown k = \frac{seq \frown k}{seq_1 \frown k}$.

(44) $(r \cdot seq) \frown k = r \cdot (seq \frown k)$.

(45) $(seq \cdot Nseq) \frown k = seq \cdot (Nseq \frown k)$.

(46) seq is a subsequence of seq.

(47) $seq \frown k$ is a subsequence of seq.

(48) If seq is a subsequence of seq_1 and seq_1 is a subsequence of seq_2, then seq is a subsequence of seq_2.

(49) If seq is increasing and seq_1 is a subsequence of seq, then seq_1 is increasing.

(50) If seq is decreasing and seq_1 is a subsequence of seq, then seq_1 is decreasing.

(51) If seq is non-decreasing and seq_1 is a subsequence of seq, then seq_1 is non-decreasing.

(52) If seq is non-increasing and seq_1 is a subsequence of seq, then seq_1 is non-increasing.

(53) If seq is monotone and seq_1 is a subsequence of seq, then seq_1 is monotone.

(54) If seq is constant and seq_1 is a subsequence of seq, then seq_1 is constant.

(55) If seq is constant and seq_1 is a subsequence of seq, then $seq = seq_1$.

(56) If seq is upper bounded and seq_1 is a subsequence of seq, then seq_1 is upper bounded.

(57) If seq is lower bounded and seq_1 is a subsequence of seq, then seq_1 is lower bounded.

(58) If seq is bounded and seq_1 is a subsequence of seq, then seq_1 is bounded.

(59) If seq is increasing and $0 < r$, then $r \cdot seq$ is increasing but if seq is increasing and $0 = r$, then $r \cdot seq$ is constant but if seq is increasing and $r < 0$, then $r \cdot seq$ is decreasing.

(60) If seq is decreasing and $0 < r$, then $r \cdot seq$ is decreasing but if seq is decreasing and $0 = r$, then $r \cdot seq$ is constant but if seq is decreasing and $r < 0$, then $r \cdot seq$ is increasing.

(61) If seq is non-decreasing and $0 \leq r$, then $r \cdot seq$ is non-decreasing but if seq is non-decreasing and $r \leq 0$, then $r \cdot seq$ is non-increasing.

(62) If seq is non-increasing and $0 \leq r$, then $r \cdot seq$ is non-increasing but if seq is non-increasing and $r \leq 0$, then $r \cdot seq$ is non-decreasing.

(63) If seq is increasing and seq_1 is increasing, then $seq + seq_1$ is increasing but if seq is decreasing and seq_1 is decreasing, then $seq+seq_1$ is decreasing but if seq is non-decreasing and seq_1 is non-decreasing, then $seq + seq_1$ is non-decreasing but if seq is non-increasing and seq_1 is non-increasing, then $seq + seq_1$ is non-increasing.

(64) If seq is increasing and seq_1 is constant, then $seq + seq_1$ is increasing but if seq is decreasing and seq_1 is constant, then $seq + seq_1$ is decreasing but if seq is non-decreasing and seq_1 is constant, then $seq + seq_1$ is non-decreasing but if seq is non-increasing and seq_1 is constant, then $seq+seq_1$ is non-increasing.

(65) If seq is constant, then for every r holds $r \cdot seq$ is constant and $-seq$ is constant and $|seq|$ is constant.

(66) If seq is constant and seq_1 is constant, then $seq \cdot seq_1$ is constant and $seq + seq_1$ is constant.

(67) If seq is constant and seq_1 is constant, then $seq - seq_1$ is constant.

(68) If seq is upper bounded and $0 < r$, then $r \cdot seq$ is upper bounded but if seq is upper bounded and $0 = r$, then $r \cdot seq$ is bounded but if seq is upper bounded and $r < 0$, then $r \cdot seq$ is lower bounded.

(69) If seq is lower bounded and $0 < r$, then $r \cdot seq$ is lower bounded but if seq is lower bounded and $0 = r$, then $r \cdot seq$ is bounded but if seq is lower bounded and $r < 0$, then $r \cdot seq$ is upper bounded.

(70) If seq is bounded, then for every r holds $r \cdot seq$ is bounded and $-seq$ is bounded and $|seq|$ is bounded.

(71) If seq is upper bounded and seq_1 is upper bounded, then $seq + seq_1$ is upper bounded but if seq is lower bounded and seq_1 is lower bounded, then $seq + seq_1$ is lower bounded but if seq is bounded and seq_1 is bounded, then $seq + seq_1$ is bounded.

(72) If seq is bounded and seq_1 is bounded, then $seq \cdot seq_1$ is bounded and $seq - seq_1$ is bounded.

(73) If seq is constant, then seq is bounded.

(74) If seq is constant, then for every r holds $r \cdot seq$ is bounded and $-seq$ is bounded and $|seq|$ is bounded.

(75) If seq is upper bounded and seq_1 is constant, then $seq + seq_1$ is upper bounded but if seq is lower bounded and seq_1 is constant, then $seq + seq_1$ is lower bounded but if seq is bounded and seq_1 is constant, then $seq + seq_1$ is bounded.

(76) If seq is upper bounded and seq_1 is constant, then $seq - seq_1$ is upper bounded but if seq is lower bounded and seq_1 is constant, then $seq - seq_1$ is lower bounded but if seq is bounded and seq_1 is constant, then $seq - seq_1$ is bounded and $seq_1 - seq$ is bounded and $seq \cdot seq_1$ is bounded.

(77) If seq is upper bounded and seq_1 is non-increasing, then $seq + seq_1$ is upper bounded.

(78) If seq is lower bounded and seq_1 is non-decreasing, then $seq + seq_1$ is lower bounded.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[3] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[4] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

Received November 23, 1989