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Fano-Desargues Parallelity Spaces [1]

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Summary. This article is the second part of Parallelity Space. It contain definition of a Fano-Desargues space, axioms of a Fano-Desargues parallelity space, definition of the relations: collinearity, parallelogram and directed congruence and some basic facts concerned with them.

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The papers [2], and [1] provide the notation and terminology for this paper. In the sequel F will denote a field. We now state a proposition

(1) Aff_{F^3} is a parallelity space.

We follow the rules: a, b, c, d, p, q, r will denote elements of the universum of Aff_{F^3}, e, f, g, h will denote elements of [: the carrier of F, the carrier of F, the carrier of F :], and K, L will denote elements of the carrier of F. One can prove the following propositions:

(2) $a, b \parallel c, d$ if and only if there exist e, f, g, h such that $\langle a, b, c, d\rangle = \langle e, f, g, h\rangle$ but there exists K such that $K \cdot (e_{\mathbf{1}} - f_{\mathbf{1}}) = g_{\mathbf{1}} - h_{\mathbf{1}}$ and $K \cdot (e_{\mathbf{2}} - f_{\mathbf{2}}) = g_{\mathbf{2}} - h_{\mathbf{2}}$ and $K \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = g_{\mathbf{3}} - h_{\mathbf{3}}$ or $e_{\mathbf{1}} - f_{\mathbf{1}} = 0_F$ and $e_{\mathbf{2}} - f_{\mathbf{2}} = 0_F$ and $e_{\mathbf{3}} - f_{\mathbf{3}} = 0_F$.

(3) If $a, b \nparallel a, c$ and $\langle a, b, a, c\rangle = \langle e, f, e, g\rangle$, then $e \neq f$ and $e \neq g$ and $f \neq g$.

(4) Suppose that
 (i) $a, b \nparallel a, c$,
 (ii) $\langle a, b, a, c\rangle = \langle e, f, e, g\rangle$,
 (iii) $K \cdot (e_{\mathbf{1}} - f_{\mathbf{1}}) = L \cdot (e_{\mathbf{1}} - g_{\mathbf{1}})$,
 (iv) $K \cdot (e_{\mathbf{2}} - f_{\mathbf{2}}) = L \cdot (e_{\mathbf{2}} - g_{\mathbf{2}})$,
 (v) $K \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = L \cdot (e_{\mathbf{3}} - g_{\mathbf{3}})$.

 Then $K = 0_F$ and $L = 0_F$.

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(5) Suppose $a, b \nparallel a, c$ and $a, b \parallel c, d$ and $a, c \parallel b, d$ and $\langle a, b, c, d \rangle = \langle e, f, g, h \rangle$. Then $h_{\mathbf{1}} = (f_{\mathbf{1}} + g_{\mathbf{1}}) - e_{\mathbf{1}}$ and $h_{\mathbf{2}} = (f_{\mathbf{2}} + g_{\mathbf{2}}) - e_{\mathbf{2}}$ and $h_{\mathbf{3}} = (f_{\mathbf{3}} + g_{\mathbf{3}}) - e_{\mathbf{3}}$.

(6) There exist a, b, c such that $a, b \nparallel a, c$.

(7) If $1_F + 1_F \neq 0_F$ and $b, c \parallel a, d$ and $a, b \parallel c, d$ and $a, c \parallel b, d$, then $a, b \parallel a, c$.

(8) If $a, p \nparallel a, b$ and $a, p \nparallel a, c$ and $a, p \parallel b, q$ and $a, p \parallel c, r$ and $a, b \parallel p, q$ and $a, c \parallel p, r$, then $b, c \parallel q, r$.

A parallelity space is called a Fano-Desarques space if:

(i) there exist elements a, b, c of the universum of it such that $a, b \nparallel a, c$,

(ii) for all elements a, b, c, d of the universum of it such that $b, c \parallel a, d$ and $a, b \parallel c, d$ and $a, c \parallel b, d$ holds $a, b \parallel a, c$,

(iii) for all elements a, b, c, p, q, r of the universum of it such that $a, p \nparallel a, b$ and $a, p \nparallel a, c$ and $a, p \parallel b, q$ and $a, p \parallel c, r$ and $a, b \parallel p, q$ and $a, c \parallel p, r$ holds $b, c \parallel q, r$.

We now state a proposition

(9) Let Fd be a parallelity space. Then the following conditions are equivalent:

  (i) there exist elements a, b, c of the universum of Fd such that $a, b \nparallel a, c$ and for all elements a, b, c, d of the universum of Fd such that $b, c \parallel a, d$ and $a, b \parallel c, d$ and $a, c \parallel b, d$ holds $a, b \parallel a, c$ and for all elements a, b, c, p, q, r of the universum of Fd such that $a, p \nparallel a, b$ and $a, p \nparallel a, c$ and $a, p \parallel b, q$ and $a, p \parallel c, r$ and $a, b \parallel p, q$ and $a, c \parallel p, r$ holds $b, c \parallel q, r$,

  (ii) Fd is a Fano-Desarques space.

We adopt the following convention: $FdSp$ is a Fano-Desarques space and a, b, c, d, p, q, r, s, o, x, y are elements of the universum of $FdSp$. The following propositions are true:

(10) There exist a, b, c such that $a, b \nparallel a, c$.

(11) If $b, c \parallel a, d$ and $a, b \parallel c, d$ and $a, c \parallel b, d$, then $a, b \parallel a, c$.

(12) If $a, p \nparallel a, b$ and $a, p \nparallel a, c$ and $a, p \parallel b, q$ and $a, p \parallel c, r$ and $a, b \parallel p, q$ and $a, c \parallel p, r$, then $b, c \parallel q, r$.

(13) If $p \neq q$, then there exists r such that $p, q \nparallel p, r$.

Let us consider $FdSp$, a, b, c. The predicate $\mathbf{L}(a, b, c)$ is defined as follows: $a, b \parallel a, c$.

The following propositions are true:

(14) $\mathbf{L}(a, b, c)$ if and only if $a, b \parallel a, c$.

(15) If $\mathbf{L}(a, b, c)$, then $\mathbf{L}(a, c, b)$ and $\mathbf{L}(c, b, a)$ and $\mathbf{L}(b, a, c)$ and $\mathbf{L}(b, c, a)$ and $\mathbf{L}(c, a, b)$.

(16) If not $\mathbf{L}(a, b, c)$, then not $\mathbf{L}(a, c, b)$ and not $\mathbf{L}(c, b, a)$ and not $\mathbf{L}(b, a, c)$ and not $\mathbf{L}(b, c, a)$ and not $\mathbf{L}(c, a, b)$.

(17) If not $\mathbf{L}(a, b, c)$ and $a, b \parallel p, q$ and $a, c \parallel p, r$ and $p \neq q$ and $p \neq r$, then not $\mathbf{L}(p, q, r)$.

(18) If $a = b$ or $b = c$ or $c = a$, then $\mathbf{L}(a, b, c)$.

(19) If $a \neq b$ and $\mathbf{L}(a, b, p)$ and $\mathbf{L}(a, b, q)$ and $\mathbf{L}(a, b, r)$, then $\mathbf{L}(p, q, r)$.

(20) If $p \neq q$, then there exists r such that not $\mathbf{L}(p, q, r)$.

(21) If $\mathbf{L}(a, b, c)$ and $\mathbf{L}(a, b, d)$, then $a, b \parallel c, d$.

(22) If not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$, then not $\mathbf{L}(a, b, d)$.

(23) If not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$ and $c \neq d$, then not $\mathbf{L}(a, b, x)$ or not $\mathbf{L}(c, d, x)$.

(24) If not $\mathbf{L}(o, a, b)$, then not $\mathbf{L}(o, a, x)$ or not $\mathbf{L}(o, b, x)$ or $o = x$.

(25) If $o \neq a$ and $o \neq b$ and $\mathbf{L}(o, a, b)$ and $\mathbf{L}(o, a, p)$ and $\mathbf{L}(o, b, q)$, then $a, b \parallel p, q$.

(26) If $a, b \nparallel c, d$ and $\mathbf{L}(a, b, p)$ and $\mathbf{L}(a, b, q)$ and $\mathbf{L}(c, d, p)$ and $\mathbf{L}(c, d, q)$, then $p = q$.

(27) If $a \neq b$ and $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$, then $a, c \parallel b, d$.

(28) If $a \neq b$ and $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$, then $c, b \parallel c, d$.

(29) If not $\mathbf{L}(o, a, c)$ and $\mathbf{L}(o, a, b)$ and $\mathbf{L}(o, c, p)$ and $\mathbf{L}(o, c, q)$ and $a, c \parallel b, p$ and $a, c \parallel b, q$, then $p = q$.

(30) If $a \neq b$ and $\mathbf{L}(a, b, c)$ and $\mathbf{L}(a, b, d)$, then $\mathbf{L}(a, c, d)$.

(31) If $\mathbf{L}(a, b, c)$ and $\mathbf{L}(a, c, d)$ and $a \neq c$, then $\mathbf{L}(b, c, d)$.

(32) $\mathbf{L}(a, b, c)$ if and only if $a, b \parallel a, c$.

Let us consider $FdSp$, a, b, c, d. The predicate $\mathbf{P}(a, b, c, d)$ is defined by: not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$ and $a, c \parallel b, d$.

Next we state a number of propositions:

(33) $\mathbf{P}(a, b, c, d)$ if and only if not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$ and $a, c \parallel b, d$.

(34) If $\mathbf{P}(a, b, c, d)$, then $a \neq b$ and $b \neq c$ and $c \neq a$ and $a \neq d$ and $b \neq d$ and $c \neq d$.

(35) If $\mathbf{P}(a, b, c, d)$, then not $\mathbf{L}(a, b, c)$ and not $\mathbf{L}(b, a, d)$ and not $\mathbf{L}(c, d, a)$ and not $\mathbf{L}(d, c, b)$.

(36) Suppose $\mathbf{P}(a, b, c, d)$. Then not $\mathbf{L}(a, b, c)$ and not $\mathbf{L}(b, a, d)$ and not $\mathbf{L}(c, d, a)$ and not $\mathbf{L}(d, c, b)$ and not $\mathbf{L}(a, c, b)$ and not $\mathbf{L}(b, a, c)$ and not $\mathbf{L}(b, c, a)$ and not $\mathbf{L}(c, a, b)$ and not $\mathbf{L}(c, b, a)$ and not $\mathbf{L}(b, d, a)$ and not $\mathbf{L}(a, b, d)$ and not $\mathbf{L}(a, d, b)$ and not $\mathbf{L}(d, a, b)$ and not $\mathbf{L}(d, b, a)$ and not $\mathbf{L}(c, a, d)$ and not $\mathbf{L}(a, c, d)$ and not $\mathbf{L}(a, d, c)$ and not $\mathbf{L}(d, a, c)$ and not $\mathbf{L}(d, c, a)$ and not $\mathbf{L}(d, b, c)$ and not $\mathbf{L}(b, c, d)$ and not $\mathbf{L}(b, d, c)$ and not $\mathbf{L}(c, b, d)$ and not $\mathbf{L}(c, d, b)$.

(37) If $\mathbf{P}(a, b, c, d)$, then not $\mathbf{L}(a, b, x)$ or not $\mathbf{L}(c, d, x)$.

(38) If $\mathbf{P}(a, b, c, d)$, then $\mathbf{P}(a, c, b, d)$.

(39) If $\mathbf{P}(a, b, c, d)$, then $\mathbf{P}(c, d, a, b)$.

(40) If $\mathbf{P}(a, b, c, d)$, then $\mathbf{P}(b, a, d, c)$.

(41) If $\mathbf{P}(a, b, c, d)$, then $\mathbf{P}(a, c, b, d)$ and $\mathbf{P}(c, d, a, b)$ and $\mathbf{P}(b, a, d, c)$ and $\mathbf{P}(c, a, d, b)$ and $\mathbf{P}(d, b, c, a)$ and $\mathbf{P}(b, d, a, c)$ and $\mathbf{P}(d, c, b, a)$.

(42) If not $\mathbf{L}(a, b, c)$, then there exists d such that $\mathbf{P}(a, b, c, d)$.

(43) If $\mathbf{P}(a, b, c, p)$ and $\mathbf{P}(a, b, c, q)$, then $p = q$.

(44) If $\mathbf{P}(a, b, c, d)$, then $a, d \nparallel b, c$.

(45) If $\mathbf{P}(a, b, c, d)$, then not $\mathbf{P}(a, b, d, c)$.

(46) If $a \neq b$, then there exists c such that $\mathbf{L}(a, b, c)$ and $c \neq a$ and $c \neq b$.

(47) If $\mathbf{P}(a, p, b, q)$ and $\mathbf{P}(a, p, c, r)$, then $b, c \parallel q, r$.

(48) If not $\mathbf{L}(b, q, c)$ and $\mathbf{P}(a, p, b, q)$ and $\mathbf{P}(a, p, c, r)$, then $\mathbf{P}(b, q, c, r)$.

(49) If $\mathbf{L}(a, b, c)$ and $b \neq c$ and $\mathbf{P}(a, p, b, q)$ and $\mathbf{P}(a, p, c, r)$, then $\mathbf{P}(b, q, c, r)$.

(50) If $\mathbf{P}(a, p, b, q)$ and $\mathbf{P}(a, p, c, r)$ and $\mathbf{P}(b, q, d, s)$, then $c, d \parallel r, s$.

(51) If $a \neq b$, then there exist c, d such that $\mathbf{P}(a, b, c, d)$.

(52) If $a \neq d$, then there exist b, c such that $\mathbf{P}(a, b, c, d)$.

(53) $\mathbf{P}(a, b, c, d)$ if and only if not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$ and $a, c \parallel b, d$.

Let us consider $FdSp$, a, b, r, s. The predicate $a, b \Rrightarrow r, s$ is defined as follows:

$a = b$ and $r = s$ or there exist p, q such that $\mathbf{P}(p, q, a, b)$ and $\mathbf{P}(p, q, r, s)$.

One can prove the following propositions:

(54) $a, b \Rrightarrow r, s$ if and only if $a = b$ and $r = s$ or there exist p, q such that $\mathbf{P}(p, q, a, b)$ and $\mathbf{P}(p, q, r, s)$.

(55) If $a, a \Rrightarrow b, c$, then $b = c$.

(56) If $a, b \Rrightarrow c, c$, then $a = b$.

(57) If $a, b \Rrightarrow b, a$, then $a = b$.

(58) If $a, b \Rrightarrow c, d$, then $a, b \parallel c, d$.

(59) If $a, b \Rrightarrow c, d$, then $a, c \parallel b, d$.

(60) If $a, b \Rrightarrow c, d$ and not $\mathbf{L}(a, b, c)$, then $\mathbf{P}(a, b, c, d)$.

(61) If $\mathbf{P}(a, b, c, d)$, then $a, b \Rrightarrow c, d$.

(62) If $a, b \Rrightarrow c, d$ and $\mathbf{L}(a, b, c)$ and $\mathbf{P}(r, s, a, b)$, then $\mathbf{P}(r, s, c, d)$.

(63) If $a, b \Rrightarrow c, x$ and $a, b \Rrightarrow c, y$, then $x = y$.

(64) There exists d such that $a, b \Rrightarrow c, d$.

(65) $a, a \Rrightarrow b, b$.

(66) $a, b \Rrightarrow a, b$.

(67) If $r, s \Rrightarrow a, b$ and $r, s \Rrightarrow c, d$, then $a, b \Rrightarrow c, d$.

(68) If $a, b \Rrightarrow c, d$, then $c, d \Rrightarrow a, b$.

(69) If $a, b \Rrightarrow c, d$, then $b, a \Rrightarrow d, c$.

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