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Introduction to Categories and Functors

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Summary. The category is introduced as an ordered 5-tuple of the form $\langle O, M, dom, cod, \cdot, id\rangle$ where O (objects) and M (morphisms) are arbitrary nonempty sets, dom and cod map M onto O and assign to a morphism domain and codomain, $\cdot$ is a partial binary map from $M \times M$ to M (composition of morphisms), id applied to an object yields the identity morphism. We define the basic notions of the category theory such as hom, monic, epi, invertible. We next define functors, the composition of functors, faithfulness and fullness of functors, isomorphism between categories and the identity functor.

MML Identifier: `CAT_1`.

The papers [5], [1], [3], [2], and [4] provide the terminology and notation for this paper. In the sequel a, b, c, o, m, x are arbitrary. Let us consider x. Then $\{x\}$ is a non-empty set.

Next we state several propositions:

(1) x is an element of $\{x\}$.

(2) For every element x of $\{a\}$ holds $x = a$.

(3) For every set X for all non-empty sets C, D for every function f from C into D for every element c of C such that $c \in X$ holds $(f \restriction X)(c) = f(c)$.

(4) For all sets X, Y, Z for every non-empty set D for every function f from X into D such that $Y \subseteq X$ and $f\,^\circ\, Y \subseteq Z$ holds $f \restriction Y$ is a function from Y into Z.

(5) For every function f from $\{a\}$ into $\{b\}$ for every element x of $\{a\}$ holds $f(x) = b$.

The arguments of the notions defined below are the following: A which is a non-empty set; b which is an object of the type reserved above. of the type reserved above. Then $A \longmapsto b$ is a function from A into $\{b\}$.

Let us consider a, b, c. The functor $\langle a, b\rangle \longmapsto c$ yields a partial function from $[\!:\{a\}, \{b\}:\!]$ to $\{c\}$ and is defined by:

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$\langle a,b\rangle \longmapsto c = \{\langle a,b\rangle\} \longmapsto c$.

One can prove the following propositions:

(6) $\langle a,b\rangle \longmapsto c = \{\langle a,b\rangle\} \longmapsto c$.

(7) $\mathrm{dom}(\langle a,b\rangle \longmapsto c) = \{\langle a,b\rangle\}$ and $\mathrm{dom}(\langle a,b\rangle \longmapsto c) = [\!:\{a\}, \{b\}:\!]$.

(8) $(\langle a,b\rangle \longmapsto c)(\langle a,b\rangle) = c$.

(9) For every element x of $\{a\}$ for every element y of $\{b\}$ holds $(\langle a,b\rangle \longmapsto c)(\langle x,y\rangle) = c$.

Let D be a non-empty set. Then id_D is a function from D into D.

We consider category structures which are systems

$\langle$ objects, morphisms, a dom-map, a cod-map, a composition, an id-map $\rangle$

where the objects, the morphisms are non-empty sets, the dom-map, the cod-map are functions from the morphisms into the objects, the composition is a partial function from $[\!:$the morphisms, the morphisms$:\!]$ to the morphisms, and the id-map is a function from the objects into the morphisms. In the sequel C denotes a category structure. We now define two new modes. Let us consider C. An object of C is an element of the objects of C.

A morphism of C is an element of the morphisms of C.

We now state two propositions:

(10) For every element a of the objects of C holds a is an object of C.

(11) For every element f of the morphisms of C holds f is a morphism of C.

We adopt the following convention: a, b, c, d are objects of C and f, g are morphisms of C. We now define two new functors. Let us consider C, f. The functor $\mathrm{dom}\, f$ yields an object of C and is defined by:

$\mathrm{dom}\, f =$(the dom-map of C)(f).

The functor $\mathrm{cod}\, f$ yielding an object of C, is defined by:

$\mathrm{cod}\, f =$(the cod-map of C)(f).

We now state two propositions:

(12) $\mathrm{dom}\, f =$(the dom-map of C)(f).

(13) $\mathrm{cod}\, f =$(the cod-map of C)(f).

Let us consider C, f, g. Let us assume that $\langle g,f\rangle \in$ dom(the composition of C). The functor $g \cdot f$ yielding a morphism of C, is defined by:

$g \cdot f =$(the composition of C)($\langle g,f\rangle$).

Next we state a proposition

(14) If $\langle g,f\rangle \in$ dom(the composition of C), then $g \cdot f =$(the composition of C)($\langle g,f\rangle$).

Let us consider C, a. The functor id_a yields a morphism of C and is defined by:

$\mathrm{id}_a =$(the id-map of C)(a).

One can prove the following proposition

(15) $\mathrm{id}_a =$(the id-map of C)(a).

Let us consider C, a, b. The functor $\mathrm{hom}(a,b)$ yielding sets of morphisms of C, is defined by:

$\hom(a, b) = \{f : \operatorname{dom} f = a \wedge \operatorname{cod} f = b\}$.

We now state four propositions:

(16) $\hom(a, b) = \{f : \operatorname{dom} f = a \wedge \operatorname{cod} f = b\}$.

(17) If $\hom(a, b) \neq \emptyset$, then there exists f such that $f \in \hom(a, b)$.

(18) $f \in \hom(a, b)$ if and only if $\operatorname{dom} f = a$ and $\operatorname{cod} f = b$.

(19) $\hom(\operatorname{dom} f, \operatorname{cod} f) \neq \emptyset$.

Let us consider C, a, b. Let us assume that $\hom(a, b) \neq \emptyset$. The mode morphism from a to b, which widens to the type a morphism of C, is defined by:

it $\in \hom(a, b)$.

Next we state several propositions:

(20) If $\hom(a, b) \neq \emptyset$, then for every morphism f of C holds f is a morphism from a to b if and only if $f \in \hom(a, b)$.

(21) For arbitrary f such that $f \in \hom(a, b)$ holds f is a morphism from a to b.

(22) For every morphism f of C holds f is a morphism from $\operatorname{dom} f$ to $\operatorname{cod} f$.

(23) For every morphism f from a to b such that $\hom(a, b) \neq \emptyset$ holds $\operatorname{dom} f = a$ and $\operatorname{cod} f = b$.

(24) For every morphism f from a to b for every morphism h from c to d such that $\hom(a, b) \neq \emptyset$ and $\hom(c, d) \neq \emptyset$ and $f = h$ holds $a = c$ and $b = d$.

(25) For every morphism f from a to b such that $\hom(a, b) = \{f\}$ for every morphism g from a to b holds $f = g$.

(26) For every morphism f from a to b such that $\hom(a, b) \neq \emptyset$ and for every morphism g from a to b holds $f = g$ holds $\hom(a, b) = \{f\}$.

(27) For every morphism f from a to b such that $\hom(a, b) \approx \hom(c, d)$ and $\hom(a, b) = \{f\}$ there exists h being a morphism from c to d such that $\hom(c, d) = \{h\}$.

The mode category, which widens to the type a category structure, is defined by:

(i) for all elements f, g of the morphisms of it holds $\langle g, f \rangle \in$ dom(the composition of it) if and only if (the dom-map of it)(g) =(the cod-map of it)(f),

(ii) for all elements f, g of the morphisms of it such that (the dom-map of it)(g) =(the cod-map of it)(f) holds (the dom-map of it)((the composition of it)($\langle g, f \rangle$)) =(the dom-map of it)(f) and (the cod-map of it)((the composition of it)($\langle g, f \rangle$)) =(the cod-map of it)(g),

(iii) for all elements f, g, h of the morphisms of it such that (the dom-map of it)(h) =(the cod-map of it)(g) and (the dom-map of it)(g) =(the cod-map of it)(f) holds (the composition of it)($\langle h$,(the composition of it)($\langle g, f \rangle$)$\rangle$) =(the composition of it)($\langle$(the composition of it)($\langle h, g \rangle$)$, f \rangle$),

(iv) for every element b of the objects of it holds (the dom-map of it)((the id-map of it)(b)) $= b$ and (the cod-map of it)((the id-map of it)(b)) $= b$ and for every element f of the morphisms of it such that (the cod-map of it)(f) $= b$ holds (the composition of it)($\langle$(the id-map of it)(b)$, f \rangle$) $= f$ and for every element g of the morphisms of it such that (the dom-map of it)$(g) = b$ holds (the composition of it)($\langle g$,(the id-map of it)$(b)\rangle) = g$.

The following three propositions are true:

(28) Let C be a category structure. Then C is a category if and only if the following conditions are satisfied:

(i) for all elements f, g of the morphisms of C holds $\langle g, f\rangle \in$ dom(the composition of C) if and only if (the dom-map of C)(g) =(the cod-map of C)(f),

(ii) for all elements f, g of the morphisms of C such that (the dom-map of C)(g) =(the cod-map of C)(f) holds (the dom-map of C)((the composition of C)($\langle g, f\rangle$)) =(the dom-map of C)(f) and (the cod-map of C)((the composition of C)($\langle g, f\rangle$)) =(the cod-map of C)(g),

(iii) for all elements f, g, h of the morphisms of C such that (the dom-map of C)(h) =(the cod-map of C)(g) and (the dom-map of C)(g) =(the cod-map of C)(f) holds (the composition of C)($\langle h$,(the composition of C)($\langle g, f\rangle$)$\rangle$) =(the composition of C)($\langle$(the composition of C)($\langle h, g\rangle$)$, f\rangle$),

(iv) for every element b of the objects of C holds (the dom-map of C)((the id-map of C)$(b)) = b$ and (the cod-map of C)((the id-map of C)$(b)) = b$ and for every element f of the morphisms of C such that (the cod-map of C)$(f) = b$ holds (the composition of C)($\langle$(the id-map of C)$(b), f\rangle) = f$ and for every element g of the morphisms of C such that (the dom-map of C)$(g) = b$ holds (the composition of C)($\langle g$,(the id-map of C)$(b)\rangle) = g$.

(29) Let C be a category structure. Suppose that

(i) for all morphisms f, g of C holds $\langle g, f\rangle \in$ dom(the composition of C) if and only if $\mathrm{dom}\, g = \mathrm{cod}\, f$,

(ii) for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\mathrm{dom}(g \cdot f) = \mathrm{dom}\, f$ and $\mathrm{cod}(g \cdot f) = \mathrm{cod}\, g$,

(iii) for all morphisms f, g, h of C such that $\mathrm{dom}\, h = \mathrm{cod}\, g$ and $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $h \cdot (g \cdot f) = (h \cdot g) \cdot f$,

(iv) for every object b of C holds $\mathrm{dom}(\mathrm{id}_b) = b$ and $\mathrm{cod}(\mathrm{id}_b) = b$ and for every morphism f of C such that $\mathrm{cod}\, f = b$ holds $\mathrm{id}_b \cdot f = f$ and for every morphism g of C such that $\mathrm{dom}\, g = b$ holds $g \cdot \mathrm{id}_b = g$.

Then C is a category.

(30) Let O, M be non-empty sets. Let d, c be functions from M into O. Let p be a partial function from $[\!:M, M:\!]$ to M. Let i be a function from O into M. Let C be a category structure. Suppose C. Then C is a category if and only if the following conditions are satisfied:

(i) for all elements f, g of M holds $\langle g, f\rangle \in \mathrm{dom}\, p$ if and only if $d(g) = c(f)$,

(ii) for all elements f, g of M such that $d(g) = c(f)$ holds $d(p(\langle g, f\rangle)) = d(f)$ and $c(p(\langle g, f\rangle)) = c(g)$,

(iii) for all elements f, g, h of M such that $d(h) = c(g)$ and $d(g) = c(f)$ holds $p(\langle h, p(\langle g, f\rangle)\rangle) = p(\langle p(\langle h, g\rangle), f\rangle)$,

(iv) for every element b of O holds $d(i(b)) = b$ and $c(i(b)) = b$ and for every element f of M such that $c(f) = b$ holds $p(\langle i(b), f\rangle) = f$ and for every element g of M such that $d(g) = b$ holds $p(\langle g, i(b)\rangle) = g$.

Let us consider o, m. The functor $\dot{\circlearrowright}(o,m)$ yielding a category, is defined by:
$\dot{\circlearrowright}(o,m) = \langle\{o\},\{m\},\{m\} \longmapsto o,\{m\} \longmapsto o,\langle m,m\rangle \longmapsto m,\{o\} \longmapsto m\rangle$.

One can prove the following propositions:

(31) $\dot{\circlearrowright}(o,m) = \langle\{o\},\{m\},\{m\} \longmapsto o,\{m\} \longmapsto o,\langle m,m\rangle \longmapsto m,\{o\} \longmapsto m\rangle$.

(32) o is an object of $\dot{\circlearrowright}(o,m)$.

(33) m is a morphism of $\dot{\circlearrowright}(o,m)$.

(34) For every object a of $\dot{\circlearrowright}(o,m)$ holds $a = o$.

(35) For every morphism f of $\dot{\circlearrowright}(o,m)$ holds $f = m$.

(36) For all objects a, b of $\dot{\circlearrowright}(o,m)$ for every morphism f of $\dot{\circlearrowright}(o,m)$ holds $f \in \hom(a,b)$.

(37) For all objects a, b of $\dot{\circlearrowright}(o,m)$ for every morphism f of $\dot{\circlearrowright}(o,m)$ holds f is a morphism from a to b.

(38) For all objects a, b of $\dot{\circlearrowright}(o,m)$ holds $\hom(a,b) \neq \emptyset$.

(39) For all objects a, b, c, d of $\dot{\circlearrowright}(o,m)$ for every morphism f from a to b for every morphism g from c to d holds $f = g$.

We adopt the following rules: B, C, D will be categories, a, b, c, d will be objects of C, and f, f_1, f_2, g, g_1, g_2 will be morphisms of C. Next we state several propositions:

(40) $\operatorname{dom} g = \operatorname{cod} f$ if and only if $\langle g,f\rangle \in \operatorname{dom}$(the composition of C).

(41) If $\operatorname{dom} g = \operatorname{cod} f$, then $g \cdot f =$(the composition of C)($\langle g,f\rangle$).

(42) For all morphisms f, g of C such that $\operatorname{dom} g = \operatorname{cod} f$ holds $\operatorname{dom}(g \cdot f) = \operatorname{dom} f$ and $\operatorname{cod}(g \cdot f) = \operatorname{cod} g$.

(43) For all morphisms f, g, h of C such that $\operatorname{dom} h = \operatorname{cod} g$ and $\operatorname{dom} g = \operatorname{cod} f$ holds $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

(44) $\operatorname{dom}(\mathrm{id}_b) = b$ and $\operatorname{cod}(\mathrm{id}_b) = b$.

(45) If $\mathrm{id}_a = \mathrm{id}_b$, then $a = b$.

(46) For every morphism f of C such that $\operatorname{cod} f = b$ holds $\mathrm{id}_b \cdot f = f$.

(47) For every morphism g of C such that $\operatorname{dom} g = b$ holds $g \cdot \mathrm{id}_b = g$.

Let us consider C, g. The predicate g is monic is defined by:
for all f_1, f_2 such that $\operatorname{dom} f_1 = \operatorname{dom} f_2$ and $\operatorname{cod} f_1 = \operatorname{dom} g$ and $\operatorname{cod} f_2 = \operatorname{dom} g$ and $g \cdot f_1 = g \cdot f_2$ holds $f_1 = f_2$.

The following proposition is true

(48) g is monic if and only if for all f_1, f_2 such that $\operatorname{dom} f_1 = \operatorname{dom} f_2$ and $\operatorname{cod} f_1 = \operatorname{dom} g$ and $\operatorname{cod} f_2 = \operatorname{dom} g$ and $g \cdot f_1 = g \cdot f_2$ holds $f_1 = f_2$.

Let us consider C, f. The predicate f is epi is defined by:
for all g_1, g_2 such that $\operatorname{dom} g_1 = \operatorname{cod} f$ and $\operatorname{dom} g_2 = \operatorname{cod} f$ and $\operatorname{cod} g_1 = \operatorname{cod} g_2$ and $g_1 \cdot f = g_2 \cdot f$ holds $g_1 = g_2$.

One can prove the following proposition

(49) f is epi if and only if for all g_1, g_2 such that $\operatorname{dom} g_1 = \operatorname{cod} f$ and $\operatorname{dom} g_2 = \operatorname{cod} f$ and $\operatorname{cod} g_1 = \operatorname{cod} g_2$ and $g_1 \cdot f = g_2 \cdot f$ holds $g_1 = g_2$.

Let us consider C, f. The predicate f is invertible is defined by:

there exists g such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ and $\mathrm{cod}\, g = \mathrm{dom}\, f$ and $f \cdot g = \mathrm{id}_{\mathrm{cod}\, f}$ and $g \cdot f = \mathrm{id}_{\mathrm{dom}\, f}$.

The following proposition is true

(50) f is invertible if and only if there exists g such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ and $\mathrm{cod}\, g = \mathrm{dom}\, f$ and $f \cdot g = \mathrm{id}_{\mathrm{cod}\, f}$ and $g \cdot f = \mathrm{id}_{\mathrm{dom}\, f}$.

In the sequel f will denote a morphism from a to b, f' will denote a morphism from b to a, g will denote a morphism from b to c, and h will denote a morphism from c to d. Next we state two propositions:

(51) If $\mathrm{hom}(a, b) \neq \emptyset$ and $\mathrm{hom}(b, c) \neq \emptyset$, then $g \cdot f \in \mathrm{hom}(a, c)$.

(52) If $\mathrm{hom}(a, b) \neq \emptyset$ and $\mathrm{hom}(b, c) \neq \emptyset$, then $\mathrm{hom}(a, c) \neq \emptyset$.

Let us consider C, a, b, c, f, g. Let us assume that $\mathrm{hom}(a, b) \neq \emptyset$ and $\mathrm{hom}(b, c) \neq \emptyset$. The functor $g \cdot f$ yields a morphism from a to c and is defined by:

$g \cdot f = g \cdot f$.

One can prove the following propositions:

(53) If $\mathrm{hom}(a, b) \neq \emptyset$ and $\mathrm{hom}(b, c) \neq \emptyset$, then $g \cdot f = g \cdot (f$ **qua** a morphism of $C)$.

(54) If $\mathrm{hom}(a, b) \neq \emptyset$ and $\mathrm{hom}(b, c) \neq \emptyset$ and $\mathrm{hom}(c, d) \neq \emptyset$, then $(h \cdot g) \cdot f = h \cdot (g \cdot f)$.

(55) $\mathrm{id}_a \in \mathrm{hom}(a, a)$.

(56) $\mathrm{hom}(a, a) \neq \emptyset$.

Let us consider C, a. Then id_a is a morphism from a to a.

The following propositions are true:

(57) If $\mathrm{hom}(a, b) \neq \emptyset$, then $\mathrm{id}_b \cdot f = f$.

(58) If $\mathrm{hom}(b, c) \neq \emptyset$, then $g \cdot \mathrm{id}_b = g$.

(59) $\mathrm{id}_a \cdot \mathrm{id}_a = \mathrm{id}_a$.

(60) If $\mathrm{hom}(b, c) \neq \emptyset$, then g is monic if and only if for every a for all morphisms f_1, f_2 from a to b such that $\mathrm{hom}(a, b) \neq \emptyset$ and $g \cdot f_1 = g \cdot f_2$ holds $f_1 = f_2$.

(61) If $\mathrm{hom}(b, c) \neq \emptyset$ and $\mathrm{hom}(c, d) \neq \emptyset$ and g is monic and h is monic, then $h \cdot g$ is monic.

(62) If $\mathrm{hom}(b, c) \neq \emptyset$ and $\mathrm{hom}(c, d) \neq \emptyset$ and $h \cdot g$ is monic, then g is monic.

(63) For every morphism h from a to b for every morphism g from b to a such that $\mathrm{hom}(a, b) \neq \emptyset$ and $\mathrm{hom}(b, a) \neq \emptyset$ and $h \cdot g = \mathrm{id}_b$ holds g is monic.

(64) id_b is monic.

(65) If $\mathrm{hom}(a, b) \neq \emptyset$, then f is epi if and only if for every c for all morphisms g_1, g_2 from b to c such that $\mathrm{hom}(b, c) \neq \emptyset$ and $g_1 \cdot f = g_2 \cdot f$ holds $g_1 = g_2$.

(66) If $\mathrm{hom}(a, b) \neq \emptyset$ and $\mathrm{hom}(b, c) \neq \emptyset$ and f is epi and g is epi, then $g \cdot f$ is epi.

(67) If $\mathrm{hom}(a, b) \neq \emptyset$ and $\mathrm{hom}(b, c) \neq \emptyset$ and $g \cdot f$ is epi, then g is epi.

(68) For every morphism h from a to b for every morphism g from b to a such that $\mathrm{hom}(a, b) \neq \emptyset$ and $\mathrm{hom}(b, a) \neq \emptyset$ and $h \cdot g = \mathrm{id}_b$ holds h is epi.

(69) id_b is epi.

(70) If $\hom(a,b) \neq \emptyset$, then f is invertible if and only if $\hom(b,a) \neq \emptyset$ and there exists g being a morphism from b to a such that $f \cdot g = \mathrm{id}_b$ and $g \cdot f = \mathrm{id}_a$.

(71) If $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$, then for all morphisms g_1, g_2 from b to a such that $f \cdot g_1 = \mathrm{id}_b$ and $g_2 \cdot f = \mathrm{id}_a$ holds $g_1 = g_2$.

Let us consider C, a, b, f. Let us assume that $\hom(a,b) \neq \emptyset$ and f is invertible. The functor f^{-1} yielding a morphism from b to a, is defined by:

$f \cdot (f^{-1}) = \mathrm{id}_b$ and $(f^{-1}) \cdot f = \mathrm{id}_a$.

We now state several propositions:

(72) If $\hom(a,b) \neq \emptyset$ and f is invertible, then for every morphism g from b to a holds $g = f^{-1}$ if and only if $f \cdot g = \mathrm{id}_b$ and $g \cdot f = \mathrm{id}_a$.

(73) If $\hom(a,b) \neq \emptyset$ and f is invertible, then f is monic and f is epi.

(74) id_a is invertible.

(75) If $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and f is invertible and g is invertible, then $g \cdot f$ is invertible.

(76) If $\hom(a,b) \neq \emptyset$ and f is invertible, then f^{-1} is invertible.

(77) If $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and f is invertible and g is invertible, then $(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$.

We now define three new predicates. Let us consider C, a. The predicate a is a terminal object is defined by:

$\hom(b,a) \neq \emptyset$ and there exists f being a morphism from b to a such that for every morphism g from b to a holds $f = g$.

The predicate a is an initial object is defined by:

$\hom(a,b) \neq \emptyset$ and there exists f being a morphism from a to b such that for every morphism g from a to b holds $f = g$.

Let us consider b. The predicate a and b are isomorphic is defined by:

$\hom(a,b) \neq \emptyset$ and there exists f such that f is invertible.

We now state a number of propositions:

(78) a is a terminal object if and only if for every b holds $\hom(b,a) \neq \emptyset$ and there exists f being a morphism from b to a such that for every morphism g from b to a holds $f = g$.

(79) a is an initial object if and only if for every b holds $\hom(a,b) \neq \emptyset$ and there exists f being a morphism from a to b such that for every morphism g from a to b holds $f = g$.

(80) a and b are isomorphic if and only if $\hom(a,b) \neq \emptyset$ and there exists f such that f is invertible.

(81) a and b are isomorphic if and only if $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$ and there exist f, f' such that $f \cdot f' = \mathrm{id}_b$ and $f' \cdot f = \mathrm{id}_a$.

(82) a is an initial object if and only if for every b there exists f being a morphism from a to b such that $\hom(a,b) = \{f\}$.

(83) If a is an initial object, then for every morphism h from a to a holds $\mathrm{id}_a = h$.

(84) If a is an initial object and b is an initial object, then a and b are isomorphic.

(85) If a is an initial object and a and b are isomorphic, then b is an initial object.

(86) b is a terminal object if and only if for every a there exists f being a morphism from a to b such that $\hom(a, b) = \{f\}$.

(87) If a is a terminal object, then for every morphism h from a to a holds $\mathrm{id}_a = h$.

(88) If a is a terminal object and b is a terminal object, then a and b are isomorphic.

(89) If b is a terminal object and a and b are isomorphic, then a is a terminal object.

(90) If $\hom(a, b) \neq \emptyset$ and a is a terminal object, then f is monic.

(91) a and a are isomorphic.

(92) If a and b are isomorphic, then b and a are isomorphic.

(93) If a and b are isomorphic and b and c are isomorphic, then a and c are isomorphic.

Let us consider C, D. The mode functor from C to D, which widens to the type a function from the morphisms of C into the morphisms of D, is defined by:

(i) for every element c of the objects of C there exists d being an element of the objects of D such that it((the id-map of C)(c)) =(the id-map of D)(d),

(ii) for every element f of the morphisms of C holds it((the id-map of C)((the dom-map of C)(f))) =(the id-map of D)((the dom-map of D)(it(f))) and it((the id-map of C)((the cod-map of C)(f))) =(the id-map of D)((the cod-map of D)(it(f))),

(iii) for all elements f, g of the morphisms of C such that $\langle g, f\rangle \in$ dom(the composition of C) holds it((the composition of C)($\langle g, f\rangle$)) =(the composition of D)($\langle$it(g), it(f)$\rangle$).

We now state two propositions:

(94) Let C, D be categories. Let T be a function from the morphisms of C into the morphisms of D. Then T is a functor from C to D if and only if the following conditions are satisfied:

(i) for every element c of the objects of C there exists d being an element of the objects of D such that T((the id-map of C)(c)) =(the id-map of D)(d),

(ii) for every element f of the morphisms of C holds T((the id-map of C)(
(the dom-map of C)(f))) =(the id-map of D)((the dom-map of D)($T(f)$)) and T((the id-map of C)((the cod-map of C)(f))) =(the id-map of D)((the cod-map of D)($T(f)$)),

(iii) for all elements f, g of the morphisms of C such that $\langle g, f\rangle \in$ dom(the composition of C) holds T((the composition of C)($\langle g, f\rangle$)) =(the composition of D)($\langle T(g), T(f)\rangle$).

(95) For all functors F_1, F_2 from C to D such that for every morphism f of C holds $F_1(f) = F_2(f)$ holds $F_1 = F_2$.

The arguments of the notions defined below are the following: C, D which are categories; F which is a function from the objects of C into the objects of D; c which is an object of C. Then $F(c)$ is an object of D.

The following propositions are true:

(96) Let T be a function from the morphisms of C into the morphisms of D. Suppose that
 (i) for every object c of C there exists d being an object of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$,
 (ii) for every morphism f of C holds $T(\mathrm{id}_{\mathrm{dom}\, f}) = \mathrm{id}_{\mathrm{dom}(T(f))}$ and $T(\mathrm{id}_{\mathrm{cod}\, f}) = \mathrm{id}_{\mathrm{cod}(T(f))}$,
 (iii) for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $T(g \cdot f) = T(g) \cdot T(f)$.
 Then T is a functor from C to D.

(97) For every functor T from C to D for every object c of C there exists d being an object of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$.

(98) For every functor T from C to D for every morphism f of C holds $T(\mathrm{id}_{\mathrm{dom}\, f}) = \mathrm{id}_{\mathrm{dom}(T(f))}$ and $T(\mathrm{id}_{\mathrm{cod}\, f}) = \mathrm{id}_{\mathrm{cod}(T(f))}$.

(99) For every functor T from C to D for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\mathrm{dom}(T(g)) = \mathrm{cod}(T(f))$ and $T(g \cdot f) = T(g) \cdot T(f)$.

(100) Let T be a function from the morphisms of C into the morphisms of D. Let F be a function from the objects of C into the objects of D. Suppose that
 (i) for every object c of C holds $T(\mathrm{id}_c) = \mathrm{id}_{F(c)}$,
 (ii) for every morphism f of C holds $F(\mathrm{dom}\, f) = \mathrm{dom}(T(f))$ and $F(\mathrm{cod}\, f) = \mathrm{cod}(T(f))$,
 (iii) for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $T(g \cdot f) = T(g) \cdot T(f)$.
 Then T is a functor from C to D.

The arguments of the notions defined below are the following: C, D which are objects of the type reserved above; F which is a function from the morphisms of C into the morphisms of D. Let us assume that for every element c of the objects of C there exists d being an element of the objects of D such that $F($(the id-map of $C)(c)) =$(the id-map of $D)(d)$. The functor $\mathrm{Obj}\, F$ yielding a function from the objects of C into the objects of D, is defined by:

for every element c of the objects of C for every element d of the objects of D such that $F($(the id-map of $C)(c)) =$(the id-map of $D)(d)$ holds $(\mathrm{Obj}\, F)(c) = d$.

Next we state several propositions:

(101) Let C, D be categories. Let T be a function from the morphisms of C into the morphisms of D. Suppose for every element c of the objects of C there exists d being an element of the objects of D such that $T($(the id-map of $C)(c)) =$(the id-map of $D)(d)$. Then for every function F from the objects of C into the objects of D holds $F = \mathrm{Obj}\, T$ if and only if for

every element c of the objects of C for every element d of the objects of D such that T((the id-map of C)(c)) =(the id-map of D)(d) holds $F(c) = d$.

(102) For every function T from the morphisms of C into the morphisms of D such that for every object c of C there exists d being an object of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$ for every object c of C for every object d of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$ holds $(\mathrm{Obj}\, T)(c) = d$.

(103) For every functor T from C to D for every object c of C for every object d of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$ holds $(\mathrm{Obj}\, T)(c) = d$.

(104) For every functor T from C to D for every object c of C holds $T(\mathrm{id}_c) = \mathrm{id}_{(\mathrm{Obj}\, T)(c)}$.

(105) For every functor T from C to D for every morphism f of C holds $(\mathrm{Obj}\, T)(\mathrm{dom}\, f) = \mathrm{dom}(T(f))$ and $(\mathrm{Obj}\, T)(\mathrm{cod}\, f) = \mathrm{cod}(T(f))$.

The arguments of the notions defined below are the following: C, D which are categories; T which is a functor from C to D; c which is an object of C. The functor $T(c)$ yielding an object of D, is defined by:

$T(c) = (\mathrm{Obj}\, T)(c)$.

We now state several propositions:

(106) For every functor T from C to D for every object c of C holds $T(c) = (\mathrm{Obj}\, T)(c)$.

(107) For every functor T from C to D for every object c of C for every object d of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$ holds $T(c) = d$.

(108) For every functor T from C to D for every object c of C holds $T(\mathrm{id}_c) = \mathrm{id}_{T(c)}$.

(109) For every functor T from C to D for every morphism f of C holds $T(\mathrm{dom}\, f) = \mathrm{dom}(T(f))$ and $T(\mathrm{cod}\, f) = \mathrm{cod}(T(f))$.

(110) For every functor T from B to C for every functor S from C to D holds $S \cdot T$ is a functor from B to D.

The arguments of the notions defined below are the following: B, C, D which are objects of the type reserved above; T which is a functor from B to C; S which is a functor from C to D. Then $S \cdot T$ is a functor from B to D.

One can prove the following three propositions:

(111) $\mathrm{id}_{\text{the morphisms of } C}$ is a functor from C to C.

(112) For every functor T from B to C for every functor S from C to D for every object b of B holds $(\mathrm{Obj}(S \cdot T))(b) = (\mathrm{Obj}\, S)((\mathrm{Obj}\, T)(b))$.

(113) For every functor T from B to C for every functor S from C to D for every object b of B holds $(S \cdot T)(b) = S(T(b))$.

Let us consider C. The functor id_C yielding a functor from C to C, is defined by:

$\mathrm{id}_C = \mathrm{id}_{\text{the morphisms of } C}$.

The following propositions are true:

(114) $\mathrm{id}_C = \mathrm{id}_{\text{the morphisms of } C}$.

(115) For every morphism f of C holds $\mathrm{id}_C(f) = f$.

(116) For every object c of C holds $(\mathrm{Obj}\,\mathrm{id}_C)(c) = c$.

(117) $\mathrm{Obj}\,\mathrm{id}_C = \mathrm{id}_{\text{the objects of } C}$.

(118) For every object c of C holds $\mathrm{id}_C(c) = c$.

We now define three new predicates. The arguments of the notions defined below are the following: C, D which are categories; T which is a functor from C to D. The predicate T is an isomorphism is defined by:

T is one-to-one and $\mathrm{rng}\,T$ =the morphisms of D and $\mathrm{rng}(\mathrm{Obj}\,T)$ =the objects of D.

The predicate T is full is defined by:

for all objects c, c' of C such that $\hom(T(c), T(c')) \neq \emptyset$ for every morphism g from $T(c)$ to $T(c')$ holds $\hom(c, c') \neq \emptyset$ and there exists f being a morphism from c to c' such that $g = T(f)$.

The predicate T is faithful is defined by:

for all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ for all morphisms f_1, f_2 from c to c' such that $T(f_1) = T(f_2)$ holds $f_1 = f_2$.

One can prove the following propositions:

(119) For every functor T from C to D holds T is an isomorphism if and only if T is one-to-one and $\mathrm{rng}\,T$ =the morphisms of D and $\mathrm{rng}(\mathrm{Obj}\,T)$ =the objects of D.

(120) For every functor T from C to D holds T is full if and only if for all objects c, c' of C such that $\hom(T(c), T(c')) \neq \emptyset$ for every morphism g from $T(c)$ to $T(c')$ holds $\hom(c, c') \neq \emptyset$ and there exists f being a morphism from c to c' such that $g = T(f)$.

(121) For every functor T from C to D holds T is faithful if and only if for all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ for all morphisms f_1, f_2 from c to c' such that $T(f_1) = T(f_2)$ holds $f_1 = f_2$.

(122) id_C is an isomorphism.

(123) For every functor T from C to D for all objects c, c' of C for arbitrary f such that $f \in \hom(c, c')$ holds $T(f) \in \hom(T(c), T(c'))$.

(124) For every functor T from C to D for all objects c,
c'
of C such that $\hom(c, c') \neq \emptyset$ for every morphism f from c to c' holds $T(f) \in \hom(T(c), T(c'))$.

(125) For every functor T from C to D for all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ for every morphism f from c to c' holds $T(f)$ is a morphism from $T(c)$ to $T(c')$.

(126) For every functor T from C to D for all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ holds $\hom(T(c), T(c')) \neq \emptyset$.

(127) For every functor T from B to C for every functor S from C to D such that T is full and S is full holds $S \cdot T$ is full.

(128) For every functor T from B to C for every functor S from C to D such that T is faithful and S is faithful holds $S \cdot T$ is faithful.

(129) For every functor T from C to D for all objects c, c' of C holds $T^\circ \hom(c,c') \subseteq \hom(T(c),T(c'))$.

The arguments of the notions defined below are the following: C, D which are categories; T which is a functor from C to D; c, c' which are objects of C. The functor $T_{c,c'}$ yielding a function from $\hom(c,c')$ into $\hom(T(c),T(c'))$, is defined by:

$T_{c,c'} = T \upharpoonright \hom(c,c')$.

One can prove the following four propositions:

(130) For every functor T from C to D for all objects c, c' of C holds $T_{c,c'} = T \upharpoonright \hom(c,c')$.

(131) For every functor T from C to D for all objects c, c' of C such that $\hom(c,c') \neq \emptyset$ for every morphism f from c to c' holds $T_{c,c'}(f) = T(f)$.

(132) For every functor T from C to D holds T is full if and only if for all objects c, c' of C holds $\operatorname{rng} T_{c,c'} = \hom(T(c),T(c'))$.

(133) For every functor T from C to D holds T is faithful if and only if for all objects c, c' of C holds $T_{c,c'}$ is one-to-one.

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