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Models and Satisfiability

Grzegorz Bancerek[1]
Warsaw University
Białystok

Summary. The article includes schemes of defining by structural induction, and definitions and theorems related to: the set of variables which have free occurrences in a ZF-formula, the set of all valuations of variables in a model, the set of all valuations which satisfy a ZF-formula in a model, the satisfiability of a ZF-formula in a model by a valuation, the validity of a ZF-formula in a model, the axioms of ZF-language, the model of the ZF set theory.

The articles [6], [7], [3], [1], [4], [5], and [2] provide the notation and terminology for this paper. For simplicity we adopt the following convention: H, H' will have the type ZF-formula; x, y, z will have the type Variable; a, b, c will have the type Any; A, X will have the type set. In the article we present several logical schemes. The scheme *ZFsch_ex* deals with a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, a unary functor $\mathcal{H}$, a binary functor $\mathcal{I}$, a binary functor $\mathcal{J}$ and a constant $\mathcal{A}$ that has the type ZF-formula, and states that the following holds

$$\mathbf{ex}\, a{,}A\ \mathbf{st}\ (\mathbf{for}\, x{,}y\ \mathbf{holds}\ \langle x \mathrel{\texttt{=}} y{,}\mathcal{F}(x,y)\rangle \in A\ \&\ \langle x \mathrel{\epsilon} y{,}\mathcal{G}(x,y)\rangle \in A)\ \&\ \langle \mathcal{A}{,}a\rangle \in A\ \&$$
$$\mathbf{for}\, H{,}a\ \mathbf{st}\ \langle H,a\rangle \in A\ \mathbf{holds}\ (H\ \text{is\_a\_equality}\ \mathbf{implies}\ a = \mathcal{F}(\mathrm{Var}_1\, H{,}\mathrm{Var}_2\, H))\ \&$$
$$(H\ \text{is\_a\_membership}\ \mathbf{implies}\ a = \mathcal{G}(\mathrm{Var}_1\, H{,}\mathrm{Var}_2\, H))\ \&$$
$$(H\ \text{is\_negative}\ \mathbf{implies}\ \mathbf{ex}\, b\ \mathbf{st}\ a = \mathcal{H}(b)\ \&\ \langle \text{the\_argument\_of}\, H{,}b\rangle \in A)\ \&$$
$$(H\ \text{is\_conjunctive}\ \mathbf{implies}\ \mathbf{ex}\, b{,}c$$
$$\mathbf{st}\ a = \mathcal{I}(b,c)\ \&\ \langle \text{the\_left\_argument\_of}\, H{,}b\rangle \in A\ \&\ \langle \text{the\_right\_argument\_of}\, H{,}c\rangle \in A)$$
$$\&\ (H\ \text{is\_universal}$$
$$\mathbf{implies}\ \mathbf{ex}\, b{,}x\ \mathbf{st}\ x = \text{bound\_in}\, H\ \&\ a = \mathcal{J}(x,b)\ \&\ \langle \text{the\_scope\_of}\, H{,}b\rangle \in A)$$

for all values of the parameters.

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The scheme *ZFsch_uniq* deals with a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, a unary functor $\mathcal{H}$, a binary functor $\mathcal{I}$, a binary functor $\mathcal{J}$, a constant $\mathcal{A}$ that has the type ZF-formula, a constant $\mathcal{B}$ and a constant $\mathcal{C}$ and states that the following holds

$$\mathcal{B} = \mathcal{C}$$

provided the parameters satisfy the following conditions:

- **ex** A **st** (**for** x,y **holds** $\langle x = y, \mathcal{F}(x, y)\rangle \in A$ & $\langle x \epsilon y, \mathcal{G}(x, y)\rangle \in A$) & $\langle \mathcal{A}, \mathcal{B}\rangle \in A$
  & **for** H,a **st** $\langle H, a\rangle \in A$ **holds**
  (H is_a_equality **implies** $a = \mathcal{F}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H)$) &
  (H is_a_membership **implies** $a = \mathcal{G}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H)$) &
  (H is_negative **implies** **ex** b **st** $a = \mathcal{H}(b)$ & $\langle$the_argument_of H,$b\rangle \in A$) &
  (H is_conjunctive **implies** **ex** b,c **st** $a = \mathcal{I}(b, c)$
  & $\langle$the_left_argument_of H,$b\rangle \in A$ & $\langle$the_right_argument_of H,$c\rangle \in A$)
  & (H is_universal
  implies **ex** b,x **st** $x =$ bound_in H & $a = \mathcal{J}(x, b)$ & $\langle$the_scope_of H,$b\rangle \in A$),
- **ex** A **st** (**for** x,y **holds** $\langle x = y, \mathcal{F}(x, y)\rangle \in A$ & $\langle x \epsilon y, \mathcal{G}(x, y)\rangle \in A$) & $\langle \mathcal{A}, \mathcal{C}\rangle \in A$
  & **for** H,a **st** $\langle H, a\rangle \in A$ **holds**
  (H is_a_equality **implies** $a = \mathcal{F}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H)$) &
  (H is_a_membership **implies** $a = \mathcal{G}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H)$) &
  (H is_negative **implies** **ex** b **st** $a = \mathcal{H}(b)$ & $\langle$the_argument_of H,$b\rangle \in A$) &
  (H is_conjunctive **implies** **ex** b,c **st** $a = \mathcal{I}(b, c)$
  & $\langle$the_left_argument_of H,$b\rangle \in A$ & $\langle$the_right_argument_of H,$c\rangle \in A$)
  & (H is_universal
  implies **ex** b,x **st** $x =$ bound_in H & $a = \mathcal{J}(x, b)$ & $\langle$the_scope_of H,$b\rangle \in A$).

The scheme *ZFsch_result* deals with a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, a unary functor $\mathcal{H}$, a binary functor $\mathcal{I}$, a binary functor $\mathcal{J}$, a constant $\mathcal{A}$ that has the type ZF-formula and a unary functor $\mathcal{K}$ and states that the following holds

($\mathcal{A}$ is_a_equality **implies** $\mathcal{K}(\mathcal{A}) = \mathcal{F}(\mathrm{Var}_1\, \mathcal{A}, \mathrm{Var}_2\, \mathcal{A})$) &
($\mathcal{A}$ is_a_membership **implies** $\mathcal{K}(\mathcal{A}) = \mathcal{G}(\mathrm{Var}_1\, \mathcal{A}, \mathrm{Var}_2\, \mathcal{A})$) &
($\mathcal{A}$ is_negative **implies** $\mathcal{K}(\mathcal{A}) = \mathcal{H}(\mathcal{K}($the_argument_of $\mathcal{A}))$) &
($\mathcal{A}$ is_conjunctive **implies** **for** a,b **st**
$a = \mathcal{K}($the_left_argument_of $\mathcal{A})$ & $b = \mathcal{K}($the_right_argument_of $\mathcal{A})$
holds $\mathcal{K}(\mathcal{A}) = \mathcal{I}(a, b)$)
& ($\mathcal{A}$ is_universal **implies** $\mathcal{K}(\mathcal{A}) = \mathcal{J}($bound_in $\mathcal{A}$,$\mathcal{K}($the_scope_of $\mathcal{A}))$)

provided the parameters satisfy the following condition:

- $$\textbf{for } H',a \textbf{ holds } a = \mathcal{K}(H') \textbf{ iff ex } A \textbf{ st}$$
$$(\textbf{for } x,y \textbf{ holds } \langle x \mathrel{=} y, \mathcal{F}(x,y)\rangle \in A \ \&\ \langle x \mathrel{\epsilon} y, \mathcal{G}(x,y)\rangle \in A) \ \&\ \langle H',a\rangle \in A \ \&$$
$$\textbf{for } H,a \textbf{ st } \langle H,a\rangle \in A \textbf{ holds } (H \text{ is\_a\_equality } \textbf{implies } a = \mathcal{F}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H))$$
$$\&\ (H \text{ is\_a\_membership } \textbf{implies } a = \mathcal{G}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H)) \ \&$$
$$(H \text{ is\_negative } \textbf{implies ex } b \textbf{ st } a = \mathcal{H}(b) \ \&\ \langle \text{the\_argument\_of } H, b\rangle \in A) \ \&$$
$$(H \text{ is\_conjunctive } \textbf{implies ex } b,c \textbf{ st } a = \mathcal{I}(b,c)$$
$$\&\ \langle \text{the\_left\_argument\_of } H, b\rangle \in A \ \&\ \langle \text{the\_right\_argument\_of } H, c\rangle \in A)$$
$$\&\ (H \text{ is\_universal}$$
$$\textbf{implies ex } b,x \textbf{ st } x = \text{bound\_in } H \ \&\ a = \mathcal{J}(x,b) \ \&\ \langle \text{the\_scope\_of } H, b\rangle \in A).$$

The scheme *ZFsch_property* concerns a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, a unary functor $\mathcal{H}$, a binary functor $\mathcal{I}$, a binary functor $\mathcal{J}$, a unary functor $\mathcal{K}$, a constant $\mathcal{A}$ that has the type ZF-formula and a unary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{P}[\mathcal{K}(\mathcal{A})]$$

provided the parameters satisfy the following conditions:

- $$\textbf{for } H',a \textbf{ holds } a = \mathcal{K}(H') \textbf{ iff ex } A \textbf{ st}$$
$$(\textbf{for } x,y \textbf{ holds } \langle x \mathrel{=} y, \mathcal{F}(x,y)\rangle \in A \ \&\ \langle x \mathrel{\epsilon} y, \mathcal{G}(x,y)\rangle \in A) \ \&\ \langle H',a\rangle \in A \ \&$$
$$\textbf{for } H,a \textbf{ st } \langle H,a\rangle \in A \textbf{ holds } (H \text{ is\_a\_equality } \textbf{implies } a = \mathcal{F}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H))$$
$$\&\ (H \text{ is\_a\_membership } \textbf{implies } a = \mathcal{G}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H)) \ \&$$
$$(H \text{ is\_negative } \textbf{implies ex } b \textbf{ st } a = \mathcal{H}(b) \ \&\ \langle \text{the\_argument\_of } H, b\rangle \in A) \ \&$$
$$(H \text{ is\_conjunctive } \textbf{implies ex } b,c \textbf{ st } a = \mathcal{I}(b,c)$$
$$\&\ \langle \text{the\_left\_argument\_of } H, b\rangle \in A \ \&\ \langle \text{the\_right\_argument\_of } H, c\rangle \in A)$$
$$\&\ (H \text{ is\_universal}$$
$$\textbf{implies ex } b,x \textbf{ st } x = \text{bound\_in } H \ \&\ a = \mathcal{J}(x,b) \ \&\ \langle \text{the\_scope\_of } H, b\rangle \in A),$$
- $$\textbf{for } x,y \textbf{ holds } \mathcal{P}[\mathcal{F}(x,y)] \ \&\ \mathcal{P}[\mathcal{G}(x,y)],$$
- $$\textbf{for } a \textbf{ st } \mathcal{P}[a] \textbf{ holds } \mathcal{P}[\mathcal{H}(a)],$$
- $$\textbf{for } a,b \textbf{ st } \mathcal{P}[a] \ \&\ \mathcal{P}[b] \textbf{ holds } \mathcal{P}[\mathcal{I}(a,b)],$$
- $$\textbf{for } a,x \textbf{ st } \mathcal{P}[a] \textbf{ holds } \mathcal{P}[\mathcal{J}(x,a)].$$

Let us consider H. The functor

$$\text{Free } H,$$

yields the type Any and is defined by

$$\textbf{ex}\,A\ \textbf{st}\ (\textbf{for}\,x,y\ \textbf{holds}\ \langle x = y,\{x,y\}\rangle \in A\ \&\ \langle x\ \epsilon\ y,\{x,y\}\rangle \in A)\ \&\ \langle H,\textbf{it}\rangle \in A\ \&$$
$$\textbf{for}\,H',a\ \textbf{st}\ \langle H',a\rangle \in A\ \textbf{holds}\ (H'\ \text{is\_a\_equality}\ \textbf{implies}\ a = \{\mathrm{Var}_1\,H',\mathrm{Var}_2\,H'\})\ \&$$
$$(H'\ \text{is\_a\_membership}\ \textbf{implies}\ a = \{\mathrm{Var}_1\,H',\mathrm{Var}_2\,H'\})\ \&$$
$$(H'\ \text{is\_negative}\ \textbf{implies}\ \textbf{ex}\,b\ \textbf{st}\ a = b\ \&\ \langle \text{the\_argument\_of}\,H',b\rangle \in A)\ \&$$
$$(H'\ \text{is\_conjunctive}\ \textbf{implies}\ \textbf{ex}\,b,c$$
$$\textbf{st}\ a = \bigcup\{b,c\}\ \&\ \langle \text{the\_left\_argument\_of}\,H',b\rangle \in A\ \&\ \langle \text{the\_right\_argument\_of}\,H',c\rangle \in A)$$
$$\&\ (H'\ \text{is\_universal}$$
$$\textbf{implies}\ \textbf{ex}\,b,x\ \textbf{st}\ x = \text{bound\_in}\,H'\ \&\ a = (\bigcup\{b\}) \setminus \{x\}\ \&\ \langle \text{the\_scope\_of}\,H',b\rangle \in A).$$

Let us consider H. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\text{Free}\,H \qquad \text{is} \qquad \textbf{set of}\ \text{Variable}\,.$$

One can prove the following proposition

(1) $$\textbf{for}\,H\ \textbf{holds}\ (H\ \text{is\_a\_equality}\ \textbf{implies}\ \text{Free}\,H = \{\mathrm{Var}_1\,H,\mathrm{Var}_2\,H\})\ \&$$
$$(H\ \text{is\_a\_membership}\ \textbf{implies}\ \text{Free}\,H = \{\mathrm{Var}_1\,H,\mathrm{Var}_2\,H\})\ \&$$
$$(H\ \text{is\_negative}\ \textbf{implies}\ \text{Free}\,H = \text{Free}\,\text{the\_argument\_of}\,H)\ \&$$
$$(H\ \text{is\_conjunctive}\ \textbf{implies}$$
$$\text{Free}\,H = \text{Free}\,\text{the\_left\_argument\_of}\,H \cup \text{Free}\,\text{the\_right\_argument\_of}\,H)$$
$$\&\ (H\ \text{is\_universal}\ \textbf{implies}\ \text{Free}\,H = (\text{Free}\,\text{the\_scope\_of}\,H) \setminus \{\text{bound\_in}\,H\}).$$

Let D have the type SET_DOMAIN. The functor

$$\text{VAL}\,D,$$

with values of the type DOMAIN, is defined by

$$a \in \textbf{it}\ \textbf{iff}\ a\ \textbf{is}\ \text{Function}\ \textbf{of}\ \text{VAR}\,,D.$$

The arguments of the notions defined below are the following: $D1$ which is an object of the type SET_DOMAIN; f which is an object of the type Function **of** VAR, $D1$; x which is an object of the type reserved above. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f\,.x \qquad \text{is} \qquad \text{Element}\ \textbf{of}\ D1.$$

For simplicity we adopt the following convention: E will denote an object of the type SET_DOMAIN; f, g will denote objects of the type Function **of** VAR, E; $v1$, $v2$, $v3$, $v4$, $v5$ will denote objects of the type Element **of** VAL E. Let us consider H, E. The functor

$$\text{St}\,(H,E),$$

yields the type Any and is defined by

$$\begin{gathered}
\textbf{ex}\, A\ \textbf{st} \\
(\textbf{for}\, x{,}y\ \textbf{holds}\ \langle x = y{,}\{\, v1 : \textbf{for}\, f\ \textbf{st}\ f = v1\ \textbf{holds}\ f\,.x = f\,.y\,\}\rangle \in A \\
\&\ \langle x \epsilon y{,}\{\, v2 : \textbf{for}\, f\ \textbf{st}\ f = v2\ \textbf{holds}\ f\,.x \in f\,.y\,\}\rangle \in A) \\
\&\ \langle H, \textbf{it}\rangle \in A\ \&\ \textbf{for}\, H'{,}a\ \textbf{st}\ \langle H'{,}a\rangle \in A\ \textbf{holds} \\
(H'\ \text{is_a_equality} \\
\textbf{implies}\ a = \{\, v3 : \textbf{for}\, f\ \textbf{st}\ f = v3\ \textbf{holds}\ f\,.(\mathrm{Var}_1\, H') = f\,.(\mathrm{Var}_2\, H')\,\}) \\
\& \\
(H'\ \text{is_a_membership} \\
\textbf{implies}\ a = \{\, v4 : \textbf{for}\, f\ \textbf{st}\ f = v4\ \textbf{holds}\ f\,.(\mathrm{Var}_1\, H') \in f\,.(\mathrm{Var}_2\, H')\,\}) \\
\&\ (H'\ \text{is_negative}\ \textbf{implies}\ \textbf{ex}\, b\ \textbf{st}\ a = (\mathrm{VAL}\, E) \setminus \bigcup\{b\}\ \&\ \langle \text{the_argument_of}\, H'{,}b\rangle \in A) \\
\& \\
(H'\ \text{is_conjunctive}\ \textbf{implies}\ \textbf{ex}\, b{,}c\ \textbf{st}\ a = (\bigcup\{b\}) \cap \bigcup\{c\} \\
\&\ \langle \text{the_left_argument_of}\, H'{,}b\rangle \in A\ \&\ \langle \text{the_right_argument_of}\, H'{,}c\rangle \in A) \\
\&\ (H'\ \text{is_universal}\ \textbf{implies}\ \textbf{ex}\, b{,}x\ \textbf{st}\ x = \text{bound_in}\, H'\ \& \\
a = \{\, v5 : \\
\textbf{for}\, X{,}f\ \textbf{st}\ X = b\ \&\ f = v5 \\
\textbf{holds}\ f \in X\ \&\ \textbf{for}\, g\ \textbf{st}\ \textbf{for}\, y\ \textbf{st}\ g.y \neq f\,.y\ \textbf{holds}\ x = y\ \textbf{holds}\ g \in X\,\} \\
\&\ \langle \text{the_scope_of}\, H'{,}b\rangle \in A).
\end{gathered}$$

Let us consider H, E. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\mathrm{St}\,(H, E) \qquad \text{is} \qquad \text{Subset}\ \textbf{of}\ \mathrm{VAL}\, E.$$

We now state a number of propositions:

$$\textbf{for}\, x{,}y{,}f\ \textbf{holds}\ f\,.x = f\,.y\ \textbf{iff}\ f \in \mathrm{St}\,(x = y{,}E), \tag{2}$$

$$\textbf{for}\, x{,}y{,}f\ \textbf{holds}\ f\,.x \in f\,.y\ \textbf{iff}\ f \in \mathrm{St}\,(x \epsilon y{,}E), \tag{3}$$

$$\textbf{for}\, H{,}f\ \textbf{holds}\ \textbf{not}\ f \in \mathrm{St}\,(H, E)\ \textbf{iff}\ f \in \mathrm{St}\,(\neg H{,}E), \tag{4}$$

$$\textbf{for}\, H{,}H'{,}f\ \textbf{holds}\ f \in \mathrm{St}\,(H, E)\ \&\ f \in \mathrm{St}\,(H'{,}E)\ \textbf{iff}\ f \in \mathrm{St}\,(H \wedge H'{,}E), \tag{5}$$

$$\begin{gathered}
\textbf{for}\, x{,}H{,}f\ \textbf{holds} \\
f \in \mathrm{St}\,(H, E)\ \&\ (\textbf{for}\, g\ \textbf{st}\ \textbf{for}\, y\ \textbf{st}\ g.y \neq f\,.y\ \textbf{holds}\ x = y\ \textbf{holds}\ g \in \mathrm{St}\,(H, E)) \\
\textbf{iff}\ f \in \mathrm{St}\,(\forall\,(x, H){,}E),
\end{gathered} \tag{6}$$

$$\begin{gathered}
H\ \text{is_a_equality} \\
\textbf{implies}\ \textbf{for}\, f\ \textbf{holds}\ f\,.(\mathrm{Var}_1\, H) = f\,.(\mathrm{Var}_2\, H)\ \textbf{iff}\ f \in \mathrm{St}\,(H, E),
\end{gathered} \tag{7}$$

$$H \text{ is\_a\_membership} \quad \textbf{implies for } f \textbf{ holds } f\,.(\mathrm{Var}_1\, H) \in f\,.(\mathrm{Var}_2\, H) \textbf{ iff } f \in \mathrm{St}\,(H, E), \tag{8}$$

$$H \text{ is\_negative} \quad \textbf{implies for } f \textbf{ holds not } f \in \mathrm{St}\,(\text{the\_argument\_of}\, H, E) \textbf{ iff } f \in \mathrm{St}\,(H, E), \tag{9}$$

$$H \text{ is\_conjunctive } \textbf{implies for } f \textbf{ holds } f \in \mathrm{St}\,(\text{the\_left\_argument\_of}\, H, E) \;\&\; f \in \mathrm{St}\,(\text{the\_right\_argument\_of}\, H, E) \textbf{ iff } f \in \mathrm{St}\,(H, E), \tag{10}$$

$$H \text{ is\_universal } \textbf{implies for } f \textbf{ holds } f \in \mathrm{St}\,(\text{the\_scope\_of}\, H, E) \;\&\; (\textbf{for } g \textbf{ st for } y \textbf{ st } g.y \neq f.y \textbf{ holds } \text{bound\_in}\, H = y \textbf{ holds } g \in \mathrm{St}\,(\text{the\_scope\_of}\, H, E)) \textbf{ iff } f \in \mathrm{St}\,(H, E). \tag{11}$$

The arguments of the notions defined below are the following: D which is an object of the type SET_DOMAIN; f which is an object of the type Function **of** VAR, D; H which is an object of the type reserved above. The predicate

$$D, f \models H \qquad \text{is defined by} \qquad f \in \mathrm{St}\,(H, D).$$

Next we state a number of propositions:

$$\textbf{for } E,f,x,y \textbf{ holds } E, f \models x = y \textbf{ iff } f.x = f.y, \tag{12}$$

$$\textbf{for } E,f,x,y \textbf{ holds } E, f \models x \,\epsilon\, y \textbf{ iff } f.x \in f.y, \tag{13}$$

$$\textbf{for } E,f,H \textbf{ holds } E, f \models H \textbf{ iff not } E, f \models \neg H, \tag{14}$$

$$\textbf{for } E,f,H,H' \textbf{ holds } E, f \models H \wedge H' \textbf{ iff } E, f \models H \;\&\; E, f \models H', \tag{15}$$

$$\textbf{for } E,f,H,x \textbf{ holds } E, f \models \forall\,(x, H) \textbf{ iff for } g \textbf{ st for } y \textbf{ st } g.y \neq f.y \textbf{ holds } x = y \textbf{ holds } E, g \models H, \tag{16}$$

$$\textbf{for } E,f,H,H' \textbf{ holds } E, f \models H \vee H' \textbf{ iff } E, f \models H \textbf{ or } E, f \models H', \tag{17}$$

$$\textbf{for } E,f,H,H' \textbf{ holds } E, f \models H \Rightarrow H' \textbf{ iff } (E, f \models H \textbf{ implies } E, f \models H'), \tag{18}$$

$$\textbf{for } E,f,H,H' \textbf{ holds } E, f \models H \Leftrightarrow H' \textbf{ iff } (E, f \models H \textbf{ iff } E, f \models H'), \tag{19}$$

$$\textbf{for } E,f,H,x \textbf{ holds } E, f \models \exists\,(x, H) \textbf{ iff ex } g \textbf{ st } (\textbf{for } y \textbf{ st } g.y \neq f.y \textbf{ holds } x = y) \;\&\; E, g \models H, \tag{20}$$

$$\textbf{for } E,f,x \quad \textbf{for } e \textbf{ being } \text{Element } \textbf{of } E \textbf{ ex } g \textbf{ st } g.x = e \;\&\; \textbf{for } z \textbf{ st } z \neq x \textbf{ holds } g.z = f.z, \tag{21}$$

(22) $$E, f \models \forall\,(x, y, H)$$

iff for g **st for** z **st** $g.z \neq f.z$ **holds** $x = z$ **or** $y = z$ **holds** $E, g \models H$,

(23) $$E, f \models \exists\,(x, y, H)$$

iff ex g **st** (**for** z **st** $g.z \neq f.z$ **holds** $x = z$ **or** $y = z$) & $E, g \models H$.

Let us consider E, H. The predicate

$E \models H$ is defined by **for** f **holds** $E, f \models H$.

One can prove the following propositions:

(24) $E \models H$ **iff for** f **holds** $E, f \models H$,

(25) $E \models \forall\,(x, H)$ **iff** $E \models H$.

We now define five new functors. The constant the_axiom_of_extensionality has the type ZF-formula, and is defined by

$$\textbf{it} = \forall\,(\xi\,0,\xi\,1,\forall\,(\xi\,2,\xi\,2 \in \xi\,0 \Leftrightarrow \xi\,2 \in \xi\,1) \Rightarrow \xi\,0 = \xi\,1).$$

The constant the_axiom_of_pairs has the type ZF-formula, and is defined by

$$\textbf{it} = \forall\,(\xi\,0,\xi\,1,\exists\,(\xi\,2,\forall\,(\xi\,3,\xi\,3 \in \xi\,2 \Leftrightarrow (\xi\,3 = \xi\,0 \vee \xi\,3 = \xi\,1)))).$$

The constant the_axiom_of_unions has the type ZF-formula, and is defined by

$$\textbf{it} = \forall\,(\xi\,0,\exists\,(\xi\,1,\forall\,(\xi\,2,\xi\,2 \in \xi\,1 \Leftrightarrow \exists\,(\xi\,3,\xi\,2 \in \xi\,3 \wedge \xi\,3 \in \xi\,0)))).$$

The constant the_axiom_of_infinity has the type ZF-formula, and is defined by

$$\textbf{it} = \exists\,(\xi\,0,$$
$$\xi\,1, \xi\,1 \in \xi\,0 \wedge \forall\,(\xi\,2,\xi\,2 \in \xi\,0 \Rightarrow \exists\,(\xi\,3,\xi\,3 \in \xi\,0 \wedge \neg\,\xi\,3 = \xi\,2 \wedge \forall\,(\xi\,4,\xi\,4 \in \xi\,2 \Rightarrow \xi\,4 \in \xi\,3)))).$$

The constant the_axiom_of_power_sets has the type ZF-formula, and is defined by

$$\textbf{it} = \forall\,(\xi\,0,\exists\,(\xi\,1,\forall\,(\xi\,2,\xi\,2 \in \xi\,1 \Leftrightarrow \forall\,(\xi\,3,\xi\,3 \in \xi\,2 \Rightarrow \xi\,3 \in \xi\,0)))).$$

Let H have the type ZF-formula. Assume that the following holds

$$\{\xi\,0,\xi\,1,\xi\,2\} \text{ misses Free}\,H.$$

The functor

the_axiom_of_substitution_for H,

with values of the type ZF-formula, is defined by

$$\textbf{it} =$$
$$\forall\,(\xi\,3,\exists\,(\xi\,0,\forall\,(\xi\,4,H \Leftrightarrow \xi\,4 = \xi\,0))) \Rightarrow \forall\,(\xi\,1,\exists\,(\xi\,2,\forall\,(\xi\,4,\xi\,4 \in \xi\,2 \Leftrightarrow \exists\,(\xi\,3,\xi\,3 \in \xi\,1 \wedge H)))).$$

We now state several propositions:

(26) $\text{the\_axiom\_of\_extensionality} = \forall(\xi\,0,\xi\,1,\forall(\xi\,2,\xi\,2 \mathrel{\epsilon} \xi\,0 \Leftrightarrow \xi\,2 \mathrel{\epsilon} \xi\,1) \Rightarrow \xi\,0 = \xi\,1)$,

(27) $\text{the\_axiom\_of\_pairs} = \forall(\xi\,0,\xi\,1,\exists(\xi\,2,\forall(\xi\,3,\xi\,3 \mathrel{\epsilon} \xi\,2 \Leftrightarrow (\xi\,3 = \xi\,0 \vee \xi\,3 = \xi\,1))))$,

(28) $\text{the\_axiom\_of\_unions} = \forall(\xi\,0,\exists(\xi\,1,\forall(\xi\,2,\xi\,2 \mathrel{\epsilon} \xi\,1 \Leftrightarrow \exists(\xi\,3,\xi\,2 \mathrel{\epsilon} \xi\,3 \wedge \xi\,3 \mathrel{\epsilon} \xi\,0))))$,

(29) $\text{the\_axiom\_of\_infinity} = \exists(\xi\,0, \xi\,1, \xi\,1 \mathrel{\epsilon} \xi\,0 \wedge \forall(\xi\,2, \xi\,2 \mathrel{\epsilon} \xi\,0 \Rightarrow \exists(\xi\,3, \xi\,3 \mathrel{\epsilon} \xi\,0 \wedge \neg\, \xi\,3 = \xi\,2 \wedge \forall(\xi\,4,\xi\,4 \mathrel{\epsilon} \xi\,2 \Rightarrow \xi\,4 \mathrel{\epsilon} \xi\,3))))$,

(30) $\text{the\_axiom\_of\_power\_sets} = \forall(\xi\,0,\exists(\xi\,1,\forall(\xi\,2,\xi\,2 \mathrel{\epsilon} \xi\,1 \Leftrightarrow \forall(\xi\,3,\xi\,3 \mathrel{\epsilon} \xi\,2 \Rightarrow \xi\,3 \mathrel{\epsilon} \xi\,0))))$,

(31) $\{\xi\,0,\xi\,1,\xi\,2\}$ misses Free H **implies** the_axiom_of_substitution_for $H =$ $\forall(\xi\,3, \exists(\xi\,0, \forall(\xi\,4, H \Leftrightarrow \xi\,4 = \xi\,0))) \Rightarrow \forall(\xi\,1,\exists(\xi\,2,\forall(\xi\,4,\xi\,4 \mathrel{\epsilon} \xi\,2 \Leftrightarrow \exists(\xi\,3,\xi\,3 \mathrel{\epsilon} \xi\,1 \wedge H))))$.

Let us consider E. The predicate

E is_a_model_of_ZF

is defined by

E is_$\in$-transitive & $E \models$ the_axiom_of_pairs & $E \models$ the_axiom_of_unions & $E \models$ the_axiom_of_infinity & $E \models$ the_axiom_of_power_sets & **for** H **st** $\{\xi\,0,\xi\,1,\xi\,2\}$ misses Free H **holds** $E \models$ the_axiom_of_substitution_for H.

The following proposition is true

(32) E is_a_model_of_ZF **iff** E is_$\in$-transitive & $E \models$ the_axiom_of_pairs & $E \models$ the_axiom_of_unions & $E \models$ the_axiom_of_infinity & $E \models$ the_axiom_of_power_sets & **for** H **st** $\{\xi\,0,\xi\,1,\xi\,2\}$ misses Free H **holds** $E \models$ the_axiom_of_substitution_for H.

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