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Tarski Grothendieck Set Theory

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Summary. This is the first part of the axiomatics of the Mizar system. It includes the axioms of the Tarski Grothendieck set theory. They are: the axiom stating that everything is a set, the extensionality axiom, the definitional axiom of the singleton, the definitional axiom of the pair, the definitional axiom of the union of a family of sets, the definitional axiom of the boolean (the power set) of a set, the regularity axiom, the definitional axiom of the ordered pair, the Tarski's axiom A introduced in [1] (see also [2]), and the Frænkel scheme. Also, the definition of equinumerosity is introduced.

MML Identifier: TARSKI.

WWW: http://mizar.org/JFM/Axiomatics/tarski.html

In this paper x, y, z, u, N, M, X, Y, Z are sets.
We now state the proposition

(2)[1] If for every x holds $x \in X$ iff $x \in Y$, then $X = Y$.

Let us consider y. The functor $\{y\}$ is defined by:

(Def. 1) $x \in \{y\}$ iff $x = y$.

Let us consider z. The functor $\{y,z\}$ is defined by:

(Def. 2) $x \in \{y,z\}$ iff $x = y$ or $x = z$.

Let us notice that the functor $\{y,z\}$ is commutative.
Let us consider X, Y. The predicate $X \subseteq Y$ is defined by:

(Def. 3) If $x \in X$, then $x \in Y$.

Let us note that the predicate $X \subseteq Y$ is reflexive.
Let us consider X. The functor $\bigcup X$ is defined by:

(Def. 4) $x \in \bigcup X$ iff there exists Y such that $x \in Y$ and $Y \in X$.

The following proposition is true

(7)[2] If $x \in X$, then there exists Y such that $Y \in X$ and it is not true that there exists x such that $x \in X$ and $x \in Y$.

[1] The proposition (1) has been removed.
[2] The propositions (3)–(6) have been removed.

The scheme *Fraenkel* deals with a set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists X such that for every x holds $x \in X$ iff there exists y such that $y \in \mathcal{A}$ and $\mathcal{P}[y,x]$

provided the parameters have the following property:

- For all x, y, z such that $\mathcal{P}[x,y]$ and $\mathcal{P}[x,z]$ holds $y = z$.

Let us consider x, y. The functor $\langle x, y\rangle$ is defined as follows:

(Def. 5) $\langle x, y\rangle = \{\{x,y\},\{x\}\}$.

Let us consider X, Y. The predicate $X \approx Y$ is defined by the condition (Def. 6).

(Def. 6) There exists Z such that

(i) for every x such that $x \in X$ there exists y such that $y \in Y$ and $\langle x, y\rangle \in Z$,

(ii) for every y such that $y \in Y$ there exists x such that $x \in X$ and $\langle x, y\rangle \in Z$, and

(iii) for all x, y, z, u such that $\langle x, y\rangle \in Z$ and $\langle z, u\rangle \in Z$ holds $x = z$ iff $y = u$.

We now state the proposition

(9)[3] There exists M such that

(i) $N \in M$,

(ii) for all X, Y such that $X \in M$ and $Y \subseteq X$ holds $Y \in M$,

(iii) for every X such that $X \in M$ there exists Z such that $Z \in M$ and for every Y such that $Y \subseteq X$ holds $Y \in Z$, and

(iv) for every X such that $X \subseteq M$ holds $X \approx M$ or $X \in M$.

References

[1] Alfred Tarski. Über Unerreichbare Kardinalzahlen. *Fundamenta Mathematicae*, 30:176–183, 1938.

[2] Alfred Tarski. On well-ordered subsets of any set. *Fundamenta Mathematicae*, 32:176–183, 1939.

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[3] The proposition (8) has been removed.