JOURNAL OF FORMALIZED MATHEMATICS
Volume 10, Released 1998, Published 2003
Inst. of Computer Science, Univ. of Białystok

Completely-Irreducible Elements[1]

Robert Milewski
University of Białystok

Summary. The article is a translation of [9, 92–93].

MML Identifier: WAYBEL16.

WWW: http://mizar.org/JFM/Vol10/waybel16.html

The articles [19], [6], [21], [16], [22], [8], [5], [18], [20], [17], [1], [2], [11], [12], [3], [7], [13], [4], [10], [14], and [15] provide the notation and terminology for this paper.

1. PRELIMINARIES

One can prove the following propositions:

(1) For every sup-semilattice L and for all elements x, y of L holds $\bigsqcap_L(\uparrow x \cap \uparrow y) = x \sqcup y$.

(2) For every semilattice L and for all elements x, y of L holds $\bigsqcup_L(\downarrow x \cap \downarrow y) = x \sqcap y$.

(3) Let L be a non empty relational structure and x, y be elements of L. If x is maximal in (the carrier of $L) \setminus \uparrow y$, then $\uparrow x \setminus \{x\} = \uparrow x \cap \uparrow y$.

(4) Let L be a non empty relational structure and x, y be elements of L. If x is minimal in (the carrier of $L) \setminus \downarrow y$, then $\downarrow x \setminus \{x\} = \downarrow x \cap \downarrow y$.

(5) Let L be a poset with l.u.b.'s, X, Y be subsets of L, and X', Y' be subsets of $L^{\rm op}$. If $X = X'$ and $Y = Y'$, then $X \sqcup Y = X' \sqcap Y'$.

(6) Let L be a poset with g.l.b.'s, X, Y be subsets of L, and X', Y' be subsets of $L^{\rm op}$. If $X = X'$ and $Y = Y'$, then $X \sqcap Y = X' \sqcup Y'$.

(7) For every non empty reflexive transitive relational structure L holds ${\rm Filt}(L) = {\rm Ids}(L^{\rm op})$.

(8) For every non empty reflexive transitive relational structure L holds ${\rm Ids}(L) = {\rm Filt}(L^{\rm op})$.

2. FREE GENERATION SET

Let S, T be complete non empty posets. A map from S into T is said to be a CLHomomorphism of S, T if:

(Def. 1) It is directed-sups-preserving and infs-preserving.

Let S be a continuous complete non empty poset and let A be a subset of S. We say that A is a free generator set if and only if the condition (Def. 2) is satisfied.

[1]This work has been supported by KBN Grant 8 T11C 018 12.

(Def. 2) Let T be a continuous complete non empty poset and f be a function from A into the carrier of T. Then there exists a CLHomomorphism h of S, T such that $h{\restriction}A = f$ and for every CLHomomorphism h' of S, T such that $h'{\restriction}A = f$ holds $h' = h$.

Let L be an upper-bounded non empty poset. Note that $\mathrm{Filt}(L)$ is non empty.

We now state a number of propositions:

(9) For every set X and for every non empty subset Y of $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle$ holds $\bigcap Y$ is a filter of $2^X_\subseteq$.

(10) For every set X and for every non empty subset Y of $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle$ holds $\inf Y$ exists in $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle$ and $\bigsqcap_{(\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle)} Y = \bigcap Y$.

(11) For every set X holds 2^X is a filter of $2^X_\subseteq$.

(12) For every set X holds $\{X\}$ is a filter of $2^X_\subseteq$.

(13) For every set X holds $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle$ is upper-bounded.

(14) For every set X holds $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle$ is lower-bounded.

(15) For every set X holds $\top_{\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle} = 2^X$.

(16) For every set X holds $\bot_{\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle} = \{X\}$.

(17) For every non empty set X and for every non empty subset Y of $\langle X, \subseteq \rangle$ such that $\sup Y$ exists in $\langle X, \subseteq \rangle$ holds $\bigcup Y \subseteq \sup Y$.

(18) For every upper-bounded semilattice L holds $\langle \mathrm{Filt}(L), \subseteq \rangle$ is complete.

Let L be an upper-bounded semilattice. Note that $\langle \mathrm{Filt}(L), \subseteq \rangle$ is complete.

3. Completely-Irreducible Elements

Let L be a non empty relational structure and let p be an element of L. We say that p is completely-irreducible if and only if:

(Def. 3) $\mathrm{Min}\, {\uparrow}p \setminus \{p\}$ exists in L.

Next we state the proposition

(19) Let L be a non empty relational structure and p be an element of L. If p is completely-irreducible, then $\bigsqcap_L({\uparrow}p \setminus \{p\}) \neq p$.

Let L be a non empty relational structure. The functor $\mathrm{Irr}\, L$ yielding a subset of L is defined as follows:

(Def. 4) For every element x of L holds $x \in \mathrm{Irr}\, L$ iff x is completely-irreducible.

We now state a number of propositions:

(20) Let L be a non empty poset and p be an element of L. Then p is completely-irreducible if and only if there exists an element q of L such that $p < q$ and for every element s of L such that $p < s$ holds $q \leq s$ and ${\uparrow}p = \{p\} \cup {\uparrow}q$.

(21) For every upper-bounded non empty poset L holds $\top_L \notin \mathrm{Irr}\, L$.

(22) For every semilattice L holds $\mathrm{Irr}\, L \subseteq \mathrm{IRR}(L)$.

(23) For every semilattice L and for every element x of L such that x is completely-irreducible holds x is irreducible.

(24) Let L be a non empty poset and x be an element of L. Suppose x is completely-irreducible. Let X be a subset of L. If $\inf X$ exists in L and $x = \inf X$, then $x \in X$.

(25) For every non empty poset L and for every subset X of L such that X is order-generating holds $\mathrm{Irr} L \subseteq X$.

(26) Let L be a complete lattice and p be an element of L. Given an element k of L such that p is maximal in (the carrier of L) $\setminus \uparrow k$. Then p is completely-irreducible.

(27) Let L be a transitive antisymmetric relational structure with l.u.b.'s and p, q, u be elements of L. Suppose $p < q$ and for every element s of L such that $p < s$ holds $q \leq s$ and $u \not\leq p$. Then $p \sqcup u = q \sqcup u$.

(28) Let L be a distributive lattice and p, q, u be elements of L. Suppose $p < q$ and for every element s of L such that $p < s$ holds $q \leq s$ and $u \not\leq p$. Then $u \sqcap q \not\leq p$.

(29) Let L be a distributive complete lattice. Suppose L^{op} is meet-continuous. Let p be an element of L. Suppose p is completely-irreducible. Then (the carrier of L) $\setminus \downarrow p$ is an open filter of L.

(30) Let L be a distributive complete lattice. Suppose L^{op} is meet-continuous. Let p be an element of L. Suppose p is completely-irreducible. Then there exists an element k of L such that $k \in$ the carrier of CompactSublatt(L) and p is maximal in (the carrier of L) $\setminus \uparrow k$.

(31) Let L be a lower-bounded algebraic lattice and x, y be elements of L. Suppose $y \not\leq x$. Then there exists an element p of L such that p is completely-irreducible and $x \leq p$ and $y \not\leq p$.

(32) Let L be a lower-bounded algebraic lattice. Then $\mathrm{Irr} L$ is order-generating and for every subset X of L such that X is order-generating holds $\mathrm{Irr} L \subseteq X$.

(33) For every lower-bounded algebraic lattice L and for every element s of L holds $s = \bigsqcap_L (\uparrow s \cap \mathrm{Irr} L)$.

(34) Let L be a complete non empty poset, X be a subset of L, and p be an element of L. If p is completely-irreducible and $p = \inf X$, then $p \in X$.

(35) Let L be a complete algebraic lattice and p be an element of L. Suppose p is completely-irreducible. Then $p = \bigsqcap_L \{x; x$ ranges over elements of L: $x \in \uparrow p \ \wedge \ \bigvee_{k:\text{element of } L} (k \in$ the carrier of CompactSublatt$(L) \ \wedge \ x$ is maximal in (the carrier of L) $\setminus \uparrow k)\}$.

(36) Let L be a complete algebraic lattice and p be an element of L. Then there exists an element k of L such that $k \in$ the carrier of CompactSublatt(L) and p is maximal in (the carrier of L) $\setminus \uparrow k$ if and only if p is completely-irreducible.

References

[1] Grzegorz Bancerek. Complete lattices. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/lattice3.html.

[2] Grzegorz Bancerek. Bounds in posets and relational substructures. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/yellow_0.html.

[3] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/waybel_0.html.

[4] Grzegorz Bancerek. The "way-below" relation. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/waybel_3.html.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[7] Czesław Byliński. Galois connections. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/waybel_1.html.

[8] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[9] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[10] Adam Grabowski. Auxiliary and approximating relations. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_4.html`.

[11] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/yellow_1.html`.

[12] Artur Korniłowicz. Definitions and properties of the join and meet of subsets. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/yellow_4.html`.

[13] Artur Korniłowicz. Meet – continuous lattices. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_2.html`.

[14] Beata Madras. Irreducible and prime elements. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_6.html`.

[15] Robert Milewski. Algebraic lattices. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_8.html`.

[16] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/setfam_1.html`.

[17] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[18] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funcop_1.html`.

[19] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[20] Wojciech A. Trybulec. Partially ordered sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/orders_1.html`.

[21] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[22] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received February 9, 1998

Published January 2, 2004