JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Vectors in Real Linear Space

Wojciech A. Trybulec
Warsaw University

Summary. In this article we introduce a notion of real linear space, operations on vectors: addition, multiplication by real number, inverse vector, subtraction. The sum of finite sequence of the vectors is also defined. Theorems that belong rather to [1] or [4] are proved.

MML Identifier: RLVECT_1.

WWW: http://mizar.org/JFM/Vol1/rlvect_1.html

The articles [11], [8], [13], [2], [3], [12], [10], [14], [6], [7], [5], [9], [4], and [1] provide the notation and terminology for this paper.

We introduce loop structures which are extensions of zero structure and are systems

$\langle$ a carrier, an addition, a zero $\rangle$,

where the carrier is a set, the addition is a binary operation on the carrier, and the zero is an element of the carrier.

We consider RLS structures as extensions of loop structure as systems

$\langle$ a carrier, a zero, an addition, an external multiplication $\rangle$,

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, and the external multiplication is a function from $[:\mathbb{R}$, the carrier$:]$ into the carrier.

Let us note that there exists an RLS structure which is non empty.

In the sequel V is a non empty RLS structure and x is a set.

Let V be an RLS structure. A vector of V is an element of V.

Let V be a 1-sorted structure and let us consider x. The predicate $x \in V$ is defined by:

(Def. 1) $x \in$ the carrier of V.

The following proposition is true

(3)[1] For every non empty 1-sorted structure V and for every element v of V holds $v \in V$.

Let V be a zero structure. The functor 0_V yielding an element of V is defined by:

(Def. 2) $0_V =$ the zero of V.

In the sequel v denotes a vector of V and a, b denote real numbers.

Let us observe that there exists a loop structure which is strict and non empty.

Let V be a non empty loop structure and let v, w be elements of V. The functor $v+w$ yields an element of V and is defined as follows:

(Def. 3) $v+w =$ (the addition of V)$(\langle v, w\rangle)$.

Let us consider V, let us consider v, and let us consider a. The functor $a \cdot v$ yields an element of V and is defined as follows:

[1] The propositions (1) and (2) have been removed.

(Def. 4) $a \cdot v =$ (the external multiplication of $V)(\langle a, v \rangle)$.

We now state the proposition

(5)[2] For every non empty loop structure V and for all elements v, w of V holds $v + w =$ (the addition of $V)(v, w)$.

Let Z_1 be a non empty set, let O be an element of Z_1, let F be a binary operation on Z_1, and let G be a function from $[:\mathbb{R}, Z_1:]$ into Z_1. Observe that $\langle Z_1, O, F, G \rangle$ is non empty.

Let I_1 be a non empty loop structure. We say that I_1 is Abelian if and only if:

(Def. 5) For all elements v, w of I_1 holds $v + w = w + v$.

We say that I_1 is add-associative if and only if:

(Def. 6) For all elements u, v, w of I_1 holds $(u + v) + w = u + (v + w)$.

We say that I_1 is right zeroed if and only if:

(Def. 7) For every element v of I_1 holds $v + 0_{(I_1)} = v$.

We say that I_1 is right complementable if and only if:

(Def. 8) For every element v of I_1 there exists an element w of I_1 such that $v + w = 0_{(I_1)}$.

Let I_1 be a non empty RLS structure. We say that I_1 is real linear space-like if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) For every a and for all vectors v, w of I_1 holds $a \cdot (v + w) = a \cdot v + a \cdot w$,

(ii) for all a, b and for every vector v of I_1 holds $(a + b) \cdot v = a \cdot v + b \cdot v$,

(iii) for all a, b and for every vector v of I_1 holds $(a \cdot b) \cdot v = a \cdot (b \cdot v)$, and

(iv) for every vector v of I_1 holds $1 \cdot v = v$.

Let us note that there exists a non empty loop structure which is strict, Abelian, add-associative, right zeroed, and right complementable.

Let us observe that there exists a non empty RLS structure which is non empty, strict, Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

A real linear space is an Abelian add-associative right zeroed right complementable real linear space-like non empty RLS structure.

Let V be an Abelian non empty loop structure and let v, w be elements of V. Let us note that the functor $v + w$ is commutative.

The following proposition is true

(7)[3] Suppose that for all vectors v, w of V holds $v + w = w + v$ and for all vectors u, v, w of V holds $(u + v) + w = u + (v + w)$ and for every vector v of V holds $v + 0_V = v$ and for every vector v of V there exists a vector w of V such that $v + w = 0_V$ and for every a and for all vectors v, w of V holds $a \cdot (v + w) = a \cdot v + a \cdot w$ and for all a, b and for every vector v of V holds $(a + b) \cdot v = a \cdot v + b \cdot v$ and for all a, b and for every vector v of V holds $(a \cdot b) \cdot v = a \cdot (b \cdot v)$ and for every vector v of V holds $1 \cdot v = v$. Then V is a real linear space.

In the sequel V is a real linear space and v, w are vectors of V.

The following proposition is true

(10)[4] Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $v + 0_V = v$ and $0_V + v = v$.

[2] The proposition (4) has been removed.
[3] The proposition (6) has been removed.
[4] The propositions (8) and (9) have been removed.

Let V be a non empty loop structure and let v be an element of V. Let us assume that V is add-associative, right zeroed, and right complementable. The functor $-v$ yields an element of V and is defined as follows:

(Def. 10) $v + -v = 0_V$.

Let V be a non empty loop structure and let v, w be elements of V. The functor $v - w$ yielding an element of V is defined as follows:

(Def. 11) $v - w = v + -w$.

We now state a number of propositions:

(16)[5] Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $v + -v = 0_V$ and $-v + v = 0_V$.

(19)[6] Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. If $v + w = 0_V$, then $v = -w$.

(20) Let V be an add-associative right zeroed right complementable non empty loop structure and v, u be elements of V. Then there exists an element w of V such that $v + w = u$.

(21) Let V be an add-associative right zeroed right complementable non empty loop structure and w, u, v_1, v_2 be elements of V. If $w + v_1 = w + v_2$ or $v_1 + w = v_2 + w$, then $v_1 = v_2$.

(22) Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. If $v + w = v$ or $w + v = v$, then $w = 0_V$.

(23) If $a = 0$ or $v = 0_V$, then $a \cdot v = 0_V$.

(24) If $a \cdot v = 0_V$, then $a = 0$ or $v = 0_V$.

(25) For every add-associative right zeroed right complementable non empty loop structure V holds $-0_V = 0_V$.

(26) Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $v - 0_V = v$.

(27) Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $0_V - v = -v$.

(28) Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $v - v = 0_V$.

(29) $-v = (-1) \cdot v$.

(30) Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $--v = v$.

(31) Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. If $-v = -w$, then $v = w$.

(33)[7] If $v = -v$, then $v = 0_V$.

(34) If $v + v = 0_V$, then $v = 0_V$.

(35) Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. If $v - w = 0_V$, then $v = w$.

(36) Let V be an add-associative right zeroed right complementable non empty loop structure and u, v be elements of V. Then there exists an element w of V such that $v - w = u$.

[5] The propositions (11)–(15) have been removed.
[6] The propositions (17) and (18) have been removed.
[7] The proposition (32) has been removed.

(37) Let V be an add-associative right zeroed right complementable non empty loop structure and w, v_1, v_2 be elements of V. If $w - v_1 = w - v_2$, then $v_1 = v_2$.

(38) $a \cdot -v = (-a) \cdot v$.

(39) $a \cdot -v = -a \cdot v$.

(40) $(-a) \cdot -v = a \cdot v$.

(41) Let V be an add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $v - (u + w) = v - w - u$.

(42) For every add-associative non empty loop structure V and for all elements v, u, w of V holds $(v + u) - w = v + (u - w)$.

(43) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $v - (u - w) = (v - u) + w$.

(44) Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. Then $-(v + w) = -w - v$.

(45) Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. Then $-(v + w) = -w + -v$.

(46) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. Then $-v - w = -w - v$.

(47) Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. Then $-(v - w) = w + -v$.

(48) $a \cdot (v - w) = a \cdot v - a \cdot w$.

(49) $(a - b) \cdot v = a \cdot v - b \cdot v$.

(50) If $a \neq 0$ and $a \cdot v = a \cdot w$, then $v = w$.

(51) If $v \neq 0_V$ and $a \cdot v = b \cdot v$, then $a = b$.

Let V be a non empty 1-sorted structure and let v, u be elements of V. Then $\langle v, u \rangle$ is a finite sequence of elements of the carrier of V.

Let V be a non empty 1-sorted structure and let v, u, w be elements of V. Then $\langle v, u, w \rangle$ is a finite sequence of elements of the carrier of V.

For simplicity, we adopt the following rules: V is a non empty loop structure, F, G are finite sequences of elements of the carrier of V, f is a function from $\mathbb{N}$ into the carrier of V, v is an element of V, and j, k, n are natural numbers.

Let us consider V and let us consider F. The functor $\sum F$ yields an element of V and is defined by:

(Def. 12) There exists f such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0_V$ and for all j, v such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$.

The following two propositions are true:

(54)[8] If $k \in \operatorname{Seg} n$ and $\operatorname{len} F = n$, then $F(k)$ is an element of V.

(55) If $\operatorname{len} F = \operatorname{len} G + 1$ and $G = F \restriction \operatorname{dom} G$ and $v = F(\operatorname{len} F)$, then $\sum F = \sum G + v$.

In the sequel V is a real linear space, v is a vector of V, and F, G are finite sequences of elements of the carrier of V.

The following three propositions are true:

[8] The propositions (52) and (53) have been removed.

(56) If $\operatorname{len} F = \operatorname{len} G$ and for all k, v such that $k \in \operatorname{dom} F$ and $v = G(k)$ holds $F(k) = a \cdot v$, then $\sum F = a \cdot \sum G$.

(57) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and F, G be finite sequences of elements of the carrier of V. Suppose $\operatorname{len} F = \operatorname{len} G$ and for every k and for every element v of V such that $k \in \operatorname{dom} F$ and $v = G(k)$ holds $F(k) = -v$. Then $\sum F = -\sum G$.

(58) Let V be an add-associative right zeroed non empty loop structure and F, G be finite sequences of elements of the carrier of V. Then $\sum(F \frown G) = \sum F + \sum G$.

For simplicity, we use the following convention: V denotes an add-associative right zeroed right complementable non empty loop structure, F denotes a finite sequence of elements of the carrier of V, v, v_1, v_2, u, w denote elements of V, and p, q denote finite sequences.

We now state a number of propositions:

(59) Let V be an Abelian add-associative right zeroed non empty loop structure and F, G be finite sequences of elements of the carrier of V. If $\operatorname{rng} F = \operatorname{rng} G$ and F is one-to-one and G is one-to-one, then $\sum F = \sum G$.

(60) For every non empty loop structure V holds $\sum(\varepsilon_{(\text{the carrier of } V)}) = 0_V$.

(61) Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $\sum\langle v\rangle = v$.

(62) Let V be an add-associative right zeroed right complementable non empty loop structure and v, u be elements of V. Then $\sum\langle v, u\rangle = v + u$.

(63) Let V be an add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle v, u, w\rangle = v + u + w$.

(64) For every real linear space V and for every real number a holds $a \cdot \sum(\varepsilon_{(\text{the carrier of } V)}) = 0_V$.

(66)[9] For every real linear space V and for every real number a and for all vectors v, u of V holds $a \cdot \sum\langle v, u\rangle = a \cdot v + a \cdot u$.

(67) Let V be a real linear space, a be a real number, and v, u, w be vectors of V. Then $a \cdot \sum\langle v, u, w\rangle = a \cdot v + a \cdot u + a \cdot w$.

(68) $-\sum(\varepsilon_{(\text{the carrier of } V)}) = 0_V$.

(69) $-\sum\langle v\rangle = -v$.

(70) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u be elements of V. Then $-\sum\langle v, u\rangle = -v - u$.

(71) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $-\sum\langle v, u, w\rangle = -v - u - w$.

(72) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. Then $\sum\langle v, w\rangle = \sum\langle w, v\rangle$.

(73) $\sum\langle v, w\rangle = \sum\langle v\rangle + \sum\langle w\rangle$.

(74) $\sum\langle 0_V, 0_V\rangle = 0_V$.

(75) $\sum\langle 0_V, v\rangle = v$ and $\sum\langle v, 0_V\rangle = v$.

(76) $\sum\langle v, -v\rangle = 0_V$ and $\sum\langle -v, v\rangle = 0_V$.

(77) $\sum\langle v, -w\rangle = v - w$.

[9] The proposition (65) has been removed.

(78) $\sum\langle -v, -w\rangle = -(w+v)$.

(79) For every real linear space V and for every vector v of V holds $\sum\langle v, v\rangle = 2 \cdot v$.

(80) For every real linear space V and for every vector v of V holds $\sum\langle -v, -v\rangle = (-2) \cdot v$.

(81) $\sum\langle u, v, w\rangle = \sum\langle u\rangle + \sum\langle v\rangle + \sum\langle w\rangle$.

(82) $\sum\langle u, v, w\rangle = \sum\langle u, v\rangle + w$.

(83) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle u, v, w\rangle = \sum\langle v, w\rangle + u$.

(84) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle u, v, w\rangle = \sum\langle u, w\rangle + v$.

(85) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle u, v, w\rangle = \sum\langle u, w, v\rangle$.

(86) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle u, v, w\rangle = \sum\langle v, u, w\rangle$.

(87) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle u, v, w\rangle = \sum\langle v, w, u\rangle$.

(89)[10] Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle u, v, w\rangle = \sum\langle w, v, u\rangle$.

(90) $\sum\langle 0_V, 0_V, 0_V\rangle = 0_V$.

(91) $\sum\langle 0_V, 0_V, v\rangle = v$ and $\sum\langle 0_V, v, 0_V\rangle = v$ and $\sum\langle v, 0_V, 0_V\rangle = v$.

(92) $\sum\langle 0_V, u, v\rangle = u + v$ and $\sum\langle u, v, 0_V\rangle = u + v$ and $\sum\langle u, 0_V, v\rangle = u + v$.

(93) For every real linear space V and for every vector v of V holds $\sum\langle v, v, v\rangle = 3 \cdot v$.

(94) If $\operatorname{len} F = 0$, then $\sum F = 0_V$.

(95) If $\operatorname{len} F = 1$, then $\sum F = F(1)$.

(96) If $\operatorname{len} F = 2$ and $v_1 = F(1)$ and $v_2 = F(2)$, then $\sum F = v_1 + v_2$.

(97) If $\operatorname{len} F = 3$ and $v_1 = F(1)$ and $v_2 = F(2)$ and $v = F(3)$, then $\sum F = v_1 + v_2 + v$.

Let R be a non empty zero structure and let a be an element of R. We say that a is non-zero if and only if:

(Def. 13) $a \neq 0_R$.

In the sequel j, k, n denote natural numbers.
One can prove the following propositions:

(98) If $j < 1$, then $j = 0$.

(99) $1 \leq k$ iff $k \neq 0$.

(102)[11] If $k \neq 0$, then $n < n + k$.

(103) $k < k + n$ iff $1 \leq n$.

(104) $\operatorname{Seg} k = \operatorname{Seg}(k+1) \setminus \{k+1\}$.

(105) $p = (p \frown q) \restriction \operatorname{dom} p$.

(106) If $\operatorname{rng} p = \operatorname{rng} q$ and p is one-to-one and q is one-to-one, then $\operatorname{len} p = \operatorname{len} q$.

[10] The proposition (88) has been removed.
[11] The propositions (100) and (101) have been removed.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[3] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal2.html.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[5] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[7] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[8] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[10] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/domain_1.html.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[12] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[13] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[14] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received July 24, 1989

Published January 2, 2004