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Inst. of Computer Science, Univ. of Białystok

Properties of Real Functions

Jarosław Kotowicz
Warsaw University
Białystok

Summary. The list of theorems concerning properties of real sequences and functions is enlarged. (See e.g. [8], [5], [10]). The monotone real functions are introduced and their properties are discussed.

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The articles [12], [14], [1], [13], [4], [15], [2], [8], [6], [5], [16], [3], [9], [11], [10], and [7] provide the notation and terminology for this paper.

For simplicity, we use the following convention: x, X, X_1, Y are sets, g, r, r_1, r_2, p are elements of $\mathbb{R}$, R is a subset of $\mathbb{R}$, s_1, s_2, s_3, s_4 are sequences of real numbers, N_1 is an increasing sequence of naturals, n is an element of $\mathbb{N}$, and h, h_1, h_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

One can prove the following propositions:

(2)[1] For all functions F, G and for every X holds $(G{\restriction}F^{\circ}X) \cdot (F{\restriction}X) = (G \cdot F){\restriction}X$.

(3) For all functions F, G and for all X, X_1 holds $(G{\restriction}X_1) \cdot (F{\restriction}X) = (G \cdot F){\restriction}(X \cap F^{-1}(X_1))$.

(4) For all functions F, G and for every X holds $X \subseteq \mathrm{dom}(G \cdot F)$ iff $X \subseteq \mathrm{dom}\,F$ and $F^{\circ}X \subseteq \mathrm{dom}\,G$.

(5) For every function F and for every X holds $(F{\restriction}X)^{\circ}X = F^{\circ}X$.

Let us consider s_1. Then $\mathrm{rng}\,s_1$ is a subset of $\mathbb{R}$.

We now state a number of propositions:

(6) $s_2 = s_3 - s_4$ iff for every n holds $s_2(n) = s_3(n) - s_4(n)$.

(7) $\mathrm{rng}(s_1 \uparrow n) \subseteq \mathrm{rng}\,s_1$.

(8) If $\mathrm{rng}\,s_1 \subseteq \mathrm{dom}\,h$, then $s_1(n) \in \mathrm{dom}\,h$.

(9) $x \in \mathrm{rng}\,s_1$ iff there exists n such that $x = s_1(n)$.

(10) $s_1(n) \in \mathrm{rng}\,s_1$.

(11) If s_2 is a subsequence of s_1, then $\mathrm{rng}\,s_2 \subseteq \mathrm{rng}\,s_1$.

(12) If s_2 is a subsequence of s_1 and s_1 is non-zero, then s_2 is non-zero.

(13) $(s_2 + s_3) \cdot N_1 = s_2 \cdot N_1 + s_3 \cdot N_1$ and $(s_2 - s_3) \cdot N_1 = s_2 \cdot N_1 - s_3 \cdot N_1$ and $(s_2\, s_3) \cdot N_1 = (s_2 \cdot N_1)\,(s_3 \cdot N_1)$.

[1] The proposition (1) has been removed.

(14) $(p\,s_1)\cdot N_1 = p\,(s_1\cdot N_1)$.

(15) $(-s_1)\cdot N_1 = -s_1\cdot N_1$ and $|s_1|\cdot N_1 = |s_1\cdot N_1|$.

(16) $(s_1\cdot N_1)^{-1} = {s_1}^{-1}\cdot N_1$.

(17) $(s_2/s_1)\cdot N_1 = (s_2\cdot N_1)/(s_1\cdot N_1)$.

(18) If s_1 is convergent and for every n holds $s_1(n) \leq 0$, then $\lim s_1 \leq 0$.

(19) If for every n holds $s_1(n) \in Y$, then $\operatorname{rng} s_1 \subseteq Y$.

Let us consider h, s_1. Let us assume that $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$. The functor $h\cdot s_1$ yielding a sequence of real numbers is defined as follows:

(Def. 1) $h\cdot s_1 = (h$ **qua** function$)\cdot(s_1)$.

One can prove the following propositions:

(21)[2] If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $(h\cdot s_1)(n) = h(s_1(n))$.

(22) If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $(h\cdot s_1)\uparrow n = h\cdot(s_1\uparrow n)$.

(23) If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1+h_2)\cdot s_1 = h_1\cdot s_1 + h_2\cdot s_1$ and $(h_1-h_2)\cdot s_1 = h_1\cdot s_1 - h_2\cdot s_1$ and $(h_1\,h_2)\cdot s_1 = (h_1\cdot s_1)\,(h_2\cdot s_1)$.

(24) For every real number r such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$ holds $(r\,h)\cdot s_1 = r\,(h\cdot s_1)$.

(25) If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $|h\cdot s_1| = |h|\cdot s_1$ and $-h\cdot s_1 = (-h)\cdot s_1$.

(26) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(\frac{1}{h})$, then $h\cdot s_1$ is non-zero.

(27) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(\frac{1}{h})$, then $\frac{1}{h}\cdot s_1 = (h\cdot s_1)^{-1}$.

(28) If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $(h\cdot s_1)\cdot N_1 = h\cdot(s_1\cdot N_1)$.

(29) If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$ and s_3 is a subsequence of s_2, then $h\cdot s_3$ is a subsequence of $h\cdot s_2$.

(30) If h is total, then $(h\cdot s_1)(n) = h(s_1(n))$.

(31) If h is total, then $h\cdot(s_1\uparrow n) = (h\cdot s_1)\uparrow n$.

(32) If h_1 is total and h_2 is total, then $(h_1+h_2)\cdot s_1 = h_1\cdot s_1 + h_2\cdot s_1$ and $(h_1-h_2)\cdot s_1 = h_1\cdot s_1 - h_2\cdot s_1$ and $(h_1\,h_2)\cdot s_1 = (h_1\cdot s_1)\,(h_2\cdot s_1)$.

(33) If h is total, then $(r\,h)\cdot s_1 = r\,(h\cdot s_1)$.

(34) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(h\restriction X)$, then $(h\restriction X)\cdot s_1 = h\cdot s_1$.

(35) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(h\restriction X)$ and if $\operatorname{rng} s_1 \subseteq \operatorname{dom}(h\restriction Y)$ or $X \subseteq Y$, then $(h\restriction X)\cdot s_1 = (h\restriction Y)\cdot s_1$.

(36) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(h\restriction X)$, then $|(h\restriction X)\cdot s_1| = (|h|\restriction X)\cdot s_1$.

(37) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(h\restriction X)$ and $h^{-1}(\{0\}) = \emptyset$, then $(\frac{1}{h}\restriction X)\cdot s_1 = ((h\restriction X)\cdot s_1)^{-1}$.

(38) If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $h^\circ \operatorname{rng} s_1 = \operatorname{rng}(h\cdot s_1)$.

(39) If $\operatorname{rng} s_1 \subseteq \operatorname{dom}(h_2\cdot h_1)$, then $h_2\cdot(h_1\cdot s_1) = (h_2\cdot h_1)\cdot s_1$.

Let Z be a set and let f be an one-to-one function. One can verify that $f\restriction Z$ is one-to-one.

Next we state three propositions:

(40) For every one-to-one function h holds $(h\restriction X)^{-1} = h^{-1}\restriction h^\circ X$.

[2] The proposition (20) has been removed.

(41) If $\operatorname{rng} h$ is bounded and $\sup \operatorname{rng} h = \inf \operatorname{rng} h$, then h is a constant on $\operatorname{dom} h$.

(42) If $Y \subseteq \operatorname{dom} h$ and $h^\circ Y$ is bounded and $\sup(h^\circ Y) = \inf(h^\circ Y)$, then h is a constant on Y.

Let us consider h, Y. We say that h is increasing on Y if and only if:

(Def. 2) For all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_1) < h(r_2)$.

We say that h is decreasing on Y if and only if:

(Def. 3) For all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_2) < h(r_1)$.

We say that h is non-decreasing on Y if and only if:

(Def. 4) For all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_1) \leq h(r_2)$.

We say that h is non increasing on Y if and only if:

(Def. 5) For all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_2) \leq h(r_1)$.

Let us consider h, Y. We say that h is monotone on Y if and only if:

(Def. 6) h is non-decreasing on Y and non increasing on Y.

We now state a number of propositions:

(48)[3] h is non-decreasing on Y iff for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 \leq r_2$ holds $h(r_1) \leq h(r_2)$.

(49) h is non increasing on Y iff for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 \leq r_2$ holds $h(r_2) \leq h(r_1)$.

(50) h is increasing on X iff $h \restriction X$ is increasing on X.

(51) h is decreasing on X iff $h \restriction X$ is decreasing on X.

(52) h is non-decreasing on X iff $h \restriction X$ is non-decreasing on X.

(53) h is non increasing on X iff $h \restriction X$ is non increasing on X.

(54) Suppose Y misses $\operatorname{dom} h$. Then h is increasing on Y, decreasing on Y, non-decreasing on Y, non increasing on Y, and monotone on Y.

(55) If h is increasing on Y, then h is non-decreasing on Y.

(56) If h is decreasing on Y, then h is non increasing on Y.

(57) If h is a constant on Y, then h is non-decreasing on Y.

(58) If h is a constant on Y, then h is non increasing on Y.

(59) If h is non-decreasing on Y and non increasing on X, then h is a constant on $Y \cap X$.

(60) If $X \subseteq Y$ and h is increasing on Y, then h is increasing on X.

(61) If $X \subseteq Y$ and h is decreasing on Y, then h is decreasing on X.

(62) If $X \subseteq Y$ and h is non-decreasing on Y, then h is non-decreasing on X.

(63) If $X \subseteq Y$ and h is non increasing on Y, then h is non increasing on X.

(64)(i) If h is increasing on Y and $0 < r$, then $r\,h$ is increasing on Y,

(ii) if $r = 0$, then $r\,h$ is a constant on Y, and

(iii) if h is increasing on Y and $r < 0$, then $r\,h$ is decreasing on Y.

[3] The propositions (43)–(47) have been removed.

(65)(i) If h is decreasing on Y and $0 < r$, then $r\,h$ is decreasing on Y, and

(ii) if h is decreasing on Y and $r < 0$, then $r\,h$ is increasing on Y.

(66)(i) If h is non-decreasing on Y and $0 \leq r$, then $r\,h$ is non-decreasing on Y, and

(ii) if h is non-decreasing on Y and $r \leq 0$, then $r\,h$ is non increasing on Y.

(67)(i) If h is non increasing on Y and $0 \leq r$, then $r\,h$ is non increasing on Y, and

(ii) if h is non increasing on Y and $r \leq 0$, then $r\,h$ is non-decreasing on Y.

(68) If $r \in X \cap Y \cap \mathrm{dom}(h_1 + h_2)$, then $r \in X \cap \mathrm{dom}\,h_1$ and $r \in Y \cap \mathrm{dom}\,h_2$.

(69)(i) If h_1 is increasing on X and h_2 is increasing on Y, then $h_1 + h_2$ is increasing on $X \cap Y$,

(ii) if h_1 is decreasing on X and h_2 is decreasing on Y, then $h_1 + h_2$ is decreasing on $X \cap Y$,

(iii) if h_1 is non-decreasing on X and h_2 is non-decreasing on Y, then $h_1 + h_2$ is non-decreasing on $X \cap Y$, and

(iv) if h_1 is non increasing on X and h_2 is non increasing on Y, then $h_1 + h_2$ is non increasing on $X \cap Y$.

(70)(i) If h_1 is increasing on X and h_2 is a constant on Y, then $h_1 + h_2$ is increasing on $X \cap Y$, and

(ii) if h_1 is decreasing on X and h_2 is a constant on Y, then $h_1 + h_2$ is decreasing on $X \cap Y$.

(71) If h_1 is increasing on X and h_2 is non-decreasing on Y, then $h_1 + h_2$ is increasing on $X \cap Y$.

(72) If h_1 is non increasing on X and h_2 is a constant on Y, then $h_1 + h_2$ is non increasing on $X \cap Y$.

(73) If h_1 is decreasing on X and h_2 is non increasing on Y, then $h_1 + h_2$ is decreasing on $X \cap Y$.

(74) If h_1 is non-decreasing on X and h_2 is a constant on Y, then $h_1 + h_2$ is non-decreasing on $X \cap Y$.

(75) h is increasing on $\{x\}$.

(76) h is decreasing on $\{x\}$.

(77) h is non-decreasing on $\{x\}$.

(78) h is non increasing on $\{x\}$.

(79) id_R is increasing on R.

(80) If h is increasing on X, then $-h$ is decreasing on X.

(81) If h is non-decreasing on X, then $-h$ is non increasing on X.

(82) If h is increasing on $[p,g]$ and decreasing on $[p,g]$, then $h\restriction[p,g]$ is one-to-one.

(83) Let h be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. If h is increasing on $[p,g]$, then $(h\restriction[p,g])^{-1}$ is increasing on $h^{\circ}[p,g]$.

(84) Let h be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. If h is decreasing on $[p,g]$, then $(h\restriction[p,g])^{-1}$ is decreasing on $h^{\circ}[p,g]$.

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