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The Fundamental Properties of Natural Numbers

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Summary. Some fundamental properties of addition, multiplication, order relations, exact division, the remainder, divisibility, the least common multiple, the greatest common divisor are presented. A proof of Euclid algorithm is also given.

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The articles [4], [6], [1], [2], [5], and [3] provide the notation and terminology for this paper.

A natural number is an element of $\mathbb{N}$.

For simplicity, we use the following convention: x is a real number, k, l, m, n are natural numbers, h, i, j are natural numbers, and X is a subset of $\mathbb{R}$.

The following proposition is true

(2)[1] For every X such that $0 \in X$ and for every x such that $x \in X$ holds $x+1 \in X$ and for every k holds $k \in X$.

Let n, k be natural numbers. Then $n+k$ is a natural number.

Let n, k be natural numbers. Note that $n+k$ is natural.

In this article we present several logical schemes. The scheme *Ind* concerns a unary predicate $\mathcal{P}$, and states that:

For every natural number k holds $\mathcal{P}[k]$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[0]$, and
- For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$.

The scheme *Nat Ind* concerns a unary predicate $\mathcal{P}$, and states that:

For every natural number k holds $\mathcal{P}[k]$

provided the following conditions are satisfied:

- $\mathcal{P}[0]$, and
- For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$.

Let n, k be natural numbers. Then $n \cdot k$ is a natural number.

Let n, k be natural numbers. Observe that $n \cdot k$ is natural.

Next we state several propositions:

(18)[2] $0 \leq i$.

(19) If $0 \neq i$, then $0 < i$.

(20) If $i \leq j$, then $i \cdot h \leq j \cdot h$.

[1] The proposition (1) has been removed.
[2] The propositions (3)–(17) have been removed.

(21) $0 \neq i+1$.

(22) $i = 0$ or there exists k such that $i = k+1$.

(23) If $i + j = 0$, then $i = 0$ and $j = 0$.

One can check that there exists a natural number which is non zero.

Let m be a natural number and let n be a non zero natural number. Observe that $m+n$ is non zero and $n+m$ is non zero.

The scheme *Def by Ind* deals with a natural number $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a natural number, and a binary predicate $\mathcal{P}$, and states that:

For every k there exists n such that $\mathcal{P}[k,n]$ and for all k, n, m such that $\mathcal{P}[k,n]$ and $\mathcal{P}[k,m]$ holds $n = m$

provided the parameters meet the following requirement:

- For all k, n holds $\mathcal{P}[k,n]$ iff $k = 0$ and $n = \mathcal{A}$ or there exist m, l such that $k = m+1$ and $\mathcal{P}[m,l]$ and $n = \mathcal{F}(k,l)$.

We now state four propositions:

(26)[3] For all i, j such that $i \leq j+1$ holds $i \leq j$ or $i = j+1$.

(27) If $i \leq j$ and $j \leq i+1$, then $i = j$ or $j = i+1$.

(28) For all i, j such that $i \leq j$ there exists k such that $j = i+k$.

(29) $i \leq i+j$.

Now we present three schemes. The scheme *Comp Ind* concerns a unary predicate $\mathcal{P}$, and states that:

For every k holds $\mathcal{P}[k]$

provided the parameters have the following property:

- For every k such that for every n such that $n < k$ holds $\mathcal{P}[n]$ holds $\mathcal{P}[k]$.

The scheme *Min* concerns a unary predicate $\mathcal{P}$, and states that:

There exists k such that $\mathcal{P}[k]$ and for every n such that $\mathcal{P}[n]$ holds $k \leq n$

provided the following requirement is met:

- There exists k such that $\mathcal{P}[k]$.

The scheme *Max* deals with a natural number $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists k such that $\mathcal{P}[k]$ and for every n such that $\mathcal{P}[n]$ holds $n \leq k$

provided the parameters meet the following requirements:

- For every k such that $\mathcal{P}[k]$ holds $k \leq \mathcal{A}$, and
- There exists k such that $\mathcal{P}[k]$.

We now state three propositions:

(37)[4] If $i \leq j$, then $i \leq j+h$.

(38) $i < j+1$ iff $i \leq j$.

(40)[5] If $i \cdot j = 1$, then $i = 1$ and $j = 1$.

The scheme *Regr* concerns a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[0]$

provided the parameters meet the following conditions:

- There exists k such that $\mathcal{P}[k]$, and
- For every k such that $k \neq 0$ and $\mathcal{P}[k]$ there exists n such that $n < k$ and $\mathcal{P}[n]$.

In the sequel t denotes a natural number.

We now state two propositions:

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[3] The propositions (24) and (25) have been removed.
[4] The propositions (30)–(36) have been removed.
[5] The proposition (39) has been removed.

(42)[6] For every m such that $0 < m$ and for every n there exist k, t such that $n = m \cdot k + t$ and $t < m$.

(43) For all natural numbers n, m, k, k_1, t, t_1 such that $n = m \cdot k + t$ and $t < m$ and $n = m \cdot k_1 + t_1$ and $t_1 < m$ holds $k = k_1$ and $t = t_1$.

Let k, l be natural numbers. The functor $k \div l$ yields a natural number and is defined by:

(Def. 1) There exists t such that $k = l \cdot (k \div l) + t$ and $t < l$ or $k \div l = 0$ and $l = 0$.

The functor $k \bmod l$ yielding a natural number is defined by:

(Def. 2) There exists t such that $k = l \cdot t + (k \bmod l)$ and $k \bmod l < l$ or $k \bmod l = 0$ and $l = 0$.

We now state two propositions:

(46)[7] If $0 < i$, then $j \bmod i < i$.

(47) If $0 < i$, then $j = i \cdot (j \div i) + (j \bmod i)$.

Let k, l be natural numbers. The predicate $k \mid l$ is defined as follows:

(Def. 3) There exists t such that $l = k \cdot t$.

Let us note that the predicate $k \mid l$ is reflexive.

We now state several propositions:

(49)[8] $j \mid i$ iff $i = j \cdot (i \div j)$.

(51)[9] If $i \mid j$ and $j \mid h$, then $i \mid h$.

(52) If $i \mid j$ and $j \mid i$, then $i = j$.

(53) $i \mid 0$ and $1 \mid i$.

(54) If $0 < j$ and $i \mid j$, then $i \leq j$.

(55) If $i \mid j$ and $i \mid h$, then $i \mid j + h$.

(56) If $i \mid j$, then $i \mid j \cdot h$.

(57) If $i \mid j$ and $i \mid j + h$, then $i \mid h$.

(58) If $i \mid j$ and $i \mid h$, then $i \mid j \bmod h$.

Let k, n be natural numbers. The functor $\mathrm{lcm}(k,n)$ yields a natural number and is defined by:

(Def. 4) $k \mid \mathrm{lcm}(k,n)$ and $n \mid \mathrm{lcm}(k,n)$ and for every m such that $k \mid m$ and $n \mid m$ holds $\mathrm{lcm}(k,n) \mid m$.

Let us observe that the functor $\mathrm{lcm}(k,n)$ is commutative and idempotent.

Let k, n be natural numbers. The functor $\gcd(k,n)$ yielding a natural number is defined as follows:

(Def. 5) $\gcd(k,n) \mid k$ and $\gcd(k,n) \mid n$ and for every m such that $m \mid k$ and $m \mid n$ holds $m \mid \gcd(k,n)$.

Let us observe that the functor $\gcd(k,n)$ is commutative and idempotent.

The scheme *Euklides* deals with a unary functor $\mathcal{F}$ yielding a natural number and natural numbers $\mathcal{A}$, $\mathcal{B}$, and states that:

There exists n such that $\mathcal{F}(n) = \gcd(\mathcal{A},\mathcal{B})$ and $\mathcal{F}(n+1) = 0$

provided the following conditions are satisfied:

- $0 < \mathcal{B}$ and $\mathcal{B} < \mathcal{A}$,
- $\mathcal{F}(0) = \mathcal{A}$ and $\mathcal{F}(1) = \mathcal{B}$, and
- For every n holds $\mathcal{F}(n+2) = \mathcal{F}(n) \bmod \mathcal{F}(n+1)$.

One can check that every natural number is ordinal.

Let us observe that there exists a subset of $\mathbb{R}$ which is non empty and ordinal.

[6] The proposition (41) has been removed.
[7] The propositions (44) and (45) have been removed.
[8] The proposition (48) has been removed.
[9] The proposition (50) has been removed.

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