JOURNAL OF FORMALIZED MATHEMATICS
Volume 5, Released 1993, Published 2003
Inst. of Computer Science, Univ. of Białystok

Representation Theorem for Boolean Algebras

Jarosław Stanisław Walijewski
Warsaw University
Białystok

MML Identifier: LOPCLSET.

WWW: http://mizar.org/JFM/Vol5/lopclset.html

The articles [13], [8], [15], [11], [16], [5], [7], [6], [4], [14], [12], [17], [2], [3], [9], [10], and [1] provide the notation and terminology for this paper.

In this paper T denotes a non empty topological space and X denotes a subset of T.

Let T be a non empty topological structure. The functor OpenClosedSet(T) yielding a family of subsets of T is defined by:

(Def. 1) OpenClosedSet$(T) = \{x; x$ ranges over subsets of T: x is open and closed$\}$.

Let T be a non empty topological space. Observe that OpenClosedSet(T) is non empty.

We now state three propositions:

(2)[1] If $X \in$ OpenClosedSet(T), then X is open.

(3) If $X \in$ OpenClosedSet(T), then X is closed.

(4) If X is open and closed, then $X \in$ OpenClosedSet(T).

In the sequel x denotes an element of OpenClosedSet(T).

Let us consider T and let C, D be elements of OpenClosedSet(T). Then $C \cup D$ is an element of OpenClosedSet(T). Then $C \cap D$ is an element of OpenClosedSet(T).

Let us consider T. The functor join(T) yields a binary operation on OpenClosedSet(T) and is defined as follows:

(Def. 2) For all elements A, B of OpenClosedSet(T) holds $(\mathrm{join}(T))(A, B) = A \cup B$.

The functor meet(T) yielding a binary operation on OpenClosedSet(T) is defined as follows:

(Def. 3) For all elements A, B of OpenClosedSet(T) holds $(\mathrm{meet}(T))(A, B) = A \cap B$.

One can prove the following propositions:

(5) For all elements x, y of $\langle \mathrm{OpenClosedSet}(T), \mathrm{join}(T), \mathrm{meet}(T) \rangle$ and for all elements x', y' of OpenClosedSet(T) such that $x = x'$ and $y = y'$ holds $x \sqcup y = x' \cup y'$.

(6) For all elements x, y of $\langle \mathrm{OpenClosedSet}(T), \mathrm{join}(T), \mathrm{meet}(T) \rangle$ and for all elements x', y' of OpenClosedSet(T) such that $x = x'$ and $y = y'$ holds $x \sqcap y = x' \cap y'$.

(7) $\emptyset_T$ is an element of OpenClosedSet(T).

[1] The proposition (1) has been removed.

(8) Ω_T is an element of OpenClosedSet(T).

(9) $x^{\rm c}$ is an element of OpenClosedSet(T).

(10) $\langle$OpenClosedSet(T), join(T), meet(T)$\rangle$ is a lattice.

Let T be a non empty topological space. The functor OpenClosedSetLatt(T) yields a lattice and is defined as follows:

(Def. 4) OpenClosedSetLatt(T) $=$ $\langle$OpenClosedSet(T), join(T), meet(T)$\rangle$.

One can prove the following propositions:

(11) For every non empty topological space T and for all elements x, y of OpenClosedSetLatt(T) holds $x \sqcup y = x \cup y$.

(12) For every non empty topological space T and for all elements x, y of OpenClosedSetLatt(T) holds $x \sqcap y = x \cap y$.

(13) The carrier of OpenClosedSetLatt(T) $=$ OpenClosedSet(T).

(14) OpenClosedSetLatt(T) is Boolean.

(15) Ω_T is an element of OpenClosedSetLatt(T).

(16) $\emptyset_T$ is an element of OpenClosedSetLatt(T).

In the sequel x, X denote sets.

Let us observe that there exists a Boolean lattice which is non trivial.

We adopt the following rules: B_1 denotes a non trivial Boolean lattice, a, b denote elements of B_1, and U_1, F denote filters of B_1.

Let us consider B_1. The functor ultraset(B_1) yields a subset of $2^{\text{the carrier of } B_1}$ and is defined as follows:

(Def. 5) ultraset(B_1) $= \{F : F$ is an ultrafilter$\}$.

Let us consider B_1. Observe that ultraset(B_1) is non empty.

The following propositions are true:

(18)[2] $x \in$ ultraset(B_1) iff there exists U_1 such that $U_1 = x$ and U_1 is an ultrafilter.

(19) For every a holds $\{F : F$ is an ultrafilter $\wedge\ a \in F\} \subseteq$ ultraset(B_1).

Let us consider B_1. The functor UFilter(B_1) yielding a function is defined by:

(Def. 6) dom UFilter(B_1) $=$ the carrier of B_1 and for every element a of B_1 holds (UFilter(B_1))(a) $=$ $\{U_1 : U_1$ is an ultrafilter $\wedge\ a \in U_1\}$.

Next we state several propositions:

(20) $x \in$ (UFilter(B_1))(a) iff there exists F such that $F = x$ and F is an ultrafilter and $a \in F$.

(21) $F \in$ (UFilter(B_1))(a) iff F is an ultrafilter and $a \in F$.

(22) For every F such that F is an ultrafilter holds $a \sqcup b \in F$ iff $a \in F$ or $b \in F$.

(23) (UFilter(B_1))($a \sqcap b$) $=$ (UFilter(B_1))(a) $\cap$ (UFilter(B_1))(b).

(24) (UFilter(B_1))($a \sqcup b$) $=$ (UFilter(B_1))(a) $\cup$ (UFilter(B_1))(b).

Let us consider B_1. Then UFilter(B_1) is a function from the carrier of B_1 into $2^{\text{ultraset}(B_1)}$.

Let us consider B_1. The functor StoneR(B_1) yields a set and is defined by:

[2] The proposition (17) has been removed.

(Def. 7) $\mathrm{StoneR}(B_1) = \mathrm{rng}\,\mathrm{UFilter}(B_1)$.

Let us consider B_1. One can check that $\mathrm{StoneR}(B_1)$ is non empty.

The following two propositions are true:

(25) $\mathrm{StoneR}(B_1) \subseteq 2^{\mathrm{ultraset}(B_1)}$.

(26) $x \in \mathrm{StoneR}(B_1)$ iff there exists a such that $(\mathrm{UFilter}(B_1))(a) = x$.

Let us consider B_1. The functor $\mathrm{StoneSpace}(B_1)$ yielding a strict topological space is defined by:

(Def. 8) The carrier of $\mathrm{StoneSpace}(B_1) = \mathrm{ultraset}(B_1)$ and the topology of $\mathrm{StoneSpace}(B_1) = \{\bigcup A; A$ ranges over families of subsets of $\mathrm{ultraset}(B_1)$: $A \subseteq \mathrm{StoneR}(B_1)\}$.

Let us consider B_1. Note that $\mathrm{StoneSpace}(B_1)$ is non empty.

We now state two propositions:

(27) If F is an ultrafilter and $F \notin (\mathrm{UFilter}(B_1))(a)$, then $a \notin F$.

(28) $\mathrm{ultraset}(B_1) \setminus (\mathrm{UFilter}(B_1))(a) = (\mathrm{UFilter}(B_1))(a^{\mathrm{c}})$.

Let us consider B_1. The functor $\mathrm{StoneBLattice}(B_1)$ yields a lattice and is defined by:

(Def. 9) $\mathrm{StoneBLattice}(B_1) = \mathrm{OpenClosedSetLatt}(\mathrm{StoneSpace}(B_1))$.

One can prove the following four propositions:

(29) $\mathrm{UFilter}(B_1)$ is one-to-one.

(30) $\bigcup \mathrm{StoneR}(B_1) = \mathrm{ultraset}(B_1)$.

(31) For all sets A, B, X such that $X \subseteq \bigcup (A \cup B)$ and for every set Y such that $Y \in B$ holds Y misses X holds $X \subseteq \bigcup A$.

(32) For every non empty set X holds there exists a finite subset of X which is non empty.

Let D be a non empty set. One can verify that there exists a finite subset of D which is non empty.

The following propositions are true:

(34)[3] Let L be a non trivial Boolean lattice and D be a non empty subset of L. Suppose $\bot_L \in [D)$. Then there exists a non empty finite subset B of the carrier of L such that $B \subseteq D$ and $\bigsqcap^{\mathrm{f}}_B = \bot_L$.

(35) For every lower bound lattice L it is not true that there exists a filter F of L such that F is an ultrafilter and $\bot_L \in F$.

(36) $(\mathrm{UFilter}(B_1))(\bot_{(B_1)}) = \emptyset$.

(37) $(\mathrm{UFilter}(B_1))(\top_{(B_1)}) = \mathrm{ultraset}(B_1)$.

(38) If $\mathrm{ultraset}(B_1) = \bigcup X$ and X is a subset of $\mathrm{StoneR}(B_1)$, then there exists a finite subset Y of X such that $\mathrm{ultraset}(B_1) = \bigcup Y$.

(40)[4] $\mathrm{StoneR}(B_1) = \mathrm{OpenClosedSet}(\mathrm{StoneSpace}(B_1))$.

Let us consider B_1. Then $\mathrm{UFilter}(B_1)$ is a homomorphism from B_1 to $\mathrm{StoneBLattice}(B_1)$.

Next we state four propositions:

(41) $\mathrm{rng}\,\mathrm{UFilter}(B_1) =$ the carrier of $\mathrm{StoneBLattice}(B_1)$.

(42) $\mathrm{UFilter}(B_1)$ is isomorphism.

(43) B_1 and $\mathrm{StoneBLattice}(B_1)$ are isomorphic.

(44) For every non trivial Boolean lattice B_1 there exists a non empty topological space T such that B_1 and $\mathrm{OpenClosedSetLatt}(T)$ are isomorphic.

---

[3] The proposition (33) has been removed.

[4] The proposition (39) has been removed.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek. Filters — part I. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/filter_0.html.

[3] Józef Białas. Group and field definitions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/realset1.html.

[4] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[6] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[7] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[8] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[9] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[10] Jolanta Kamieńska and Jarosław Stanisław Walijewski. Homomorphisms of lattices, finite join and finite meet. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/lattice4.html.

[11] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[12] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[14] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finsub_1.html.

[15] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[16] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[17] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/lattices.html.

Received July 14, 1993

Published January 2, 2004