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Full Trees

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The articles [19], [10], [22], [21], [20], [2], [17], [23], [1], [24], [18], [8], [9], [13], [6], [11], [12], [16], [15], [3], [4], [5], [7], and [14] provide the notation and terminology for this paper.

1. TREES AND BINARY TREES

One can prove the following two propositions:

(1) For every set D and for every finite sequence p and for every natural number n such that $p \in D^*$ holds $p \restriction \mathrm{Seg}\, n \in D^*$.

(2) For every binary tree T holds every element of T is a finite sequence of elements of *Boolean*.

Let T be a binary tree. We see that the element of T is a finite sequence of elements of *Boolean*.
One can prove the following propositions:

(3) For every tree T such that $T = \{0,1\}^*$ holds T is binary.

(4) For every tree T such that $T = \{0,1\}^*$ holds $\mathrm{Leaves}(T) = \emptyset$.

(5) Let T be a binary tree, n be a natural number, and t be an element of T. If $t \in T\text{-level}(n)$, then t is a n-tuple of *Boolean*.

(6) For every tree T such that for every element t of T holds $\mathrm{succ}\, t = \{t \frown \langle 0 \rangle, t \frown \langle 1 \rangle\}$ holds $\mathrm{Leaves}(T) = \emptyset$.

(7) For every tree T such that for every element t of T holds $\mathrm{succ}\, t = \{t \frown \langle 0 \rangle, t \frown \langle 1 \rangle\}$ holds T is binary.

(8) For every tree T holds $T = \{0,1\}^*$ iff for every element t of T holds $\mathrm{succ}\, t = \{t \frown \langle 0 \rangle, t \frown \langle 1 \rangle\}$.

In this article we present several logical schemes. The scheme *DecoratedBinTreeEx* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

There exists a binary tree D decorated with elements of $\mathcal{A}$ such that $\mathrm{dom}\, D = \{0,1\}^*$ and $D(\emptyset) = \mathcal{B}$ and for every node x of D holds $\mathcal{P}[D(x), D(x \frown \langle 0 \rangle), D(x \frown \langle 1 \rangle)]$

provided the following condition is met:

- For every element a of $\mathcal{A}$ there exist elements b, c of $\mathcal{A}$ such that $\mathcal{P}[a,b,c]$.

The scheme *DecoratedBinTreeEx1* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

There exists a binary tree D decorated with elements of $\mathcal{A}$ such that $\mathrm{dom}\,D = \{0,1\}^*$ and $D(\emptyset) = \mathcal{B}$ and for every node x of D holds $\mathcal{P}[D(x), D(x ^\frown \langle 0 \rangle)]$ and $Q[D(x), D(x ^\frown \langle 1 \rangle)]$

provided the parameters meet the following conditions:

- For every element a of $\mathcal{A}$ there exists an element b of $\mathcal{A}$ such that $\mathcal{P}[a,b]$, and
- For every element a of $\mathcal{A}$ there exists an element b of $\mathcal{A}$ such that $Q[a,b]$.

Let T be a binary tree and let n be a non empty natural number. The functor $\mathrm{NumberOnLevel}(n,T)$ yields a function from T-level(n) into $\mathbb{N}$ and is defined as follows:

(Def. 1) For every element t of T such that $t \in T$-level(n) and for every n-tuple F of *Boolean* such that $F = \mathrm{Rev}(t)$ holds $(\mathrm{NumberOnLevel}(n,T))(t) = \mathrm{Absval}(F) + 1$.

Let T be a binary tree and let n be a non empty natural number. One can check that $\mathrm{NumberOnLevel}(n,T)$ is one-to-one.

2. Full Trees

Let T be a tree. We say that T is full if and only if:

(Def. 2) $T = \{0,1\}^*$.

We now state three propositions:

(9) $\{0,1\}^*$ is a tree.

(10) For every tree T such that $T = \{0,1\}^*$ and for every natural number n holds $\langle \underbrace{0,\ldots,0}_{n} \rangle \in T$-level$(n)$.

(11) For every tree T such that $T = \{0,1\}^*$ and for every non empty natural number n and for every n-tuple y of *Boolean* holds $y \in T$-level(n).

Let T be a binary tree and let n be a natural number. Observe that T-level(n) is finite.

Let us observe that every tree which is full is also binary.

Let us note that there exists a tree which is full.

The following proposition is true

(12) For every full tree T and for every non empty natural number n holds $\mathrm{Seg}(2^n) \subseteq \mathrm{rng}\,\mathrm{NumberOnLevel}(n,T)$.

Let T be a full tree and let n be a non empty natural number. The functor $\mathrm{FinSeqLevel}(n,T)$ yields a finite sequence of elements of T-level(n) and is defined by:

(Def. 3) $\mathrm{FinSeqLevel}(n,T) = (\mathrm{NumberOnLevel}(n,T))^{-1}$.

Let T be a full tree and let n be a non empty natural number. Observe that $\mathrm{FinSeqLevel}(n,T)$ is one-to-one.

Next we state a number of propositions:

(13) For every full tree T and for every non empty natural number n holds $(\mathrm{NumberOnLevel}(n,T))(\langle \underbrace{0,\ldots,0}_{n} \rangle) = 1$.

(14) Let T be a full tree, n be a non empty natural number, and y be a n-tuple of *Boolean*. If $y = \langle \underbrace{0,\ldots,0}_{n} \rangle$, then $(\mathrm{NumberOnLevel}(n,T))(\neg y) = 2^n$.

(15) For every full tree T and for every non empty natural number n holds $(\mathrm{FinSeqLevel}(n,T))(1) = \langle \underbrace{0,\ldots,0}_{n} \rangle$.

(16) Let T be a full tree, n be a non empty natural number, and y be a n-tuple of *Boolean*. If $y = \langle \underbrace{0,\ldots,0}_{n} \rangle$, then $(\mathrm{FinSeqLevel}(n,T))(2^n) = \neg y$.

(17) Let T be a full tree, n be a non empty natural number, and i be a natural number. If $i \in \mathrm{Seg}(2^n)$, then $(\mathrm{FinSeqLevel}(n,T))(i) = \mathrm{Rev}(n\text{-BinarySequence}(i -' 1))$.

(18) For every full tree T and for every natural number n holds $\overline{\overline{T\text{-level}(n)}} = 2^n$.

(19) For every full tree T and for every non empty natural number n holds $\mathrm{len}\,\mathrm{FinSeqLevel}(n,T) = 2^n$.

(20) For every full tree T and for every non empty natural number n holds $\mathrm{dom}\,\mathrm{FinSeqLevel}(n,T) = \mathrm{Seg}(2^n)$.

(21) For every full tree T and for every non empty natural number n holds $\mathrm{rng}\,\mathrm{FinSeqLevel}(n,T) = T\text{-level}(n)$.

(22) For every full tree T holds $(\mathrm{FinSeqLevel}(1,T))(1) = \langle 0 \rangle$.

(23) For every full tree T holds $(\mathrm{FinSeqLevel}(1,T))(2) = \langle 1 \rangle$.

(24) Let T be a full tree and n, i be non empty natural numbers. Suppose $i \leq 2^{n+1}$. Let F be a n-tuple of *Boolean*. If $F = (\mathrm{FinSeqLevel}(n,T))((i+1) \div 2)$, then $(\mathrm{FinSeqLevel}(n+1,T))(i) = F \frown \langle (i+1) \bmod 2 \rangle$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[3] Grzegorz Bancerek. Introduction to trees. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/trees_1.html.

[4] Grzegorz Bancerek. König's Lemma. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/trees_2.html.

[5] Grzegorz Bancerek. Joining of decorated trees. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/trees_4.html.

[6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[7] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on binary trees. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/bintree1.html.

[8] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[9] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[10] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[11] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[12] Czesław Byliński. Some properties of restrictions of finite sequences. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/finseq_5.html.

[13] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[14] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/euclid.html.

[15] Robert Milewski. Binary arithmetics. Binary sequences. *Journal of Formalized Mathematics*, 10, 1998. http://mizar.org/JFM/Vol10/binari_3.html.

[16] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/binarith.html.

[17] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/series_1.html.

[18] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/domain_1.html`.

[19] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[20] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/mcart_1.html`.

[21] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[22] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[23] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[24] Edmund Woronowicz. Many-argument relations. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/margrel1.html`.

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